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UNITARY SPHERICAL SPECTRUM FOR SPLIT CLASSICAL GROUPS

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1. Introduction

This paper gives a classification of the spherical dual of the split groups $Sp(n)$ and $So(n)$ over the real and p-adic field. Most of the results were known earlier from [B1], [B2], [B3] and [BM3]. As is explained in these references, in the p-adic case the classification of the spherical unitary dual is equivalent to the classification of the unitary generic Iwahori-spherical modules. The new result is the proof of necessary conditions for unitarity in the real case. Following a suggestion of D. Vogan, I find a set of K-types which I call *relevant* which detect the nonunitarity. They have the property that they are in 1-1 correspondence with certain irreducible Weyl group representations so that the intertwining operators are *the same* in the real and p-adic case. Thus the same proof applies in both cases. Since the answer is independent of the field, this establishes a form of the Lefschetz principle.

Let G be a split symplectic or orthogonal group over a local field $\mathbb{F}$ which is either $\mathbb{R}$ or a p-adic field. In general, for a group H, we will denote by $\mathfrak{h}$ its Lie algebra. Fix a maximal compact subgroup K. In the real case, there is only one conjugacy class. In the p-adic case, $\mathbb{F} \supset \mathcal{R} \supset \mathcal{P}$, where $\mathcal{R}$ is the ring of integers and $\mathcal{P}$ the maximal prime ideal. We fix $K = G(\mathcal{R})$. Fix also a rational Borel subgroup $B = AN$. Then $G = KB$, and denote by $M := K \cap B$. A representation (π, V) (admissible) is called spherical if $V^K \neq (0)$. The classification of irreducible admissible spherical modules is well known. For every irreducible spherical π, there is a character $\chi \in \widehat{A}$ such that $\chi|_{A\cap K} = triv$, and π is the unique spherical subquotient of $Ind_B^G[\chi \otimes 1\!\!1]$. We will call a character χ whose restriction to $A \cap K$ is trivial, *unramified.* Since G is split, $A \cong (\mathbb{F}^\times)^n$ where n is the rank. Thus an unramified character is of the form

$$\chi(a_1, \dots, a_n) = |a_1|^{\nu_1} \dots |a_n|^{\nu_n}, \qquad \nu_i \in \mathbb{C}/[(2i\pi/\log q)\mathbb{Z}], \qquad (1.0.1)$$

where q is the order of $\mathcal{R}/\mathcal{P}$. Thus (essentially) each unramified character is determined by an element in $X^*(A) \otimes_{\mathbb{Z}} \mathbb{C}$. We call this element χ as well. Write $X(\chi)$ for the induced module (principal series) and $L(\chi)$ for the irreducible spherical subquotient. Two such modules $L(\chi)$ and $L(\chi')$ are equivalent if and only if there is an element in the Weyl group W such that

$w\chi = \chi'$. An $L(\chi)$ is hermitian if and only if there is w such that $w\chi = -\overline{\chi}$. The paper deals exclusively with what is called in [BM] *real* infinitesimal character. In the real case it is well known that any representation is unitarily induced irreducible from a representation with real infinitesimal character on a proper Levi component. In the p-adic case, the results in [BM2] show that the problem of classifying the unitary dual reduces to determining the unitary dual with real infinitesimal character of a Hecke algebra. This Hecke algebra is not necessarily one for a Levi component of the group. This is why we assume that $\chi \in \mathfrak{a}^*$, the (real) dual of the Lie algebra $\mathfrak{a}$.

Let $\check{G}$ be the (complex) dual group, and let $\check{A}$ be the torus dual to A. Then we can interpret χ as an element of $\check{\mathfrak{a}}$, or more generally a conjugacy class of a real semisimple element of $\check{\mathfrak{g}}$. We attach to each χ a nilpotent orbit satisfying the following properties. Fix a Lie triple $\{\check{e}, \check{h}, \check{f}\}$ corresponding to $\check{\mathcal{O}}$ such that $\check{h} \in \check{\mathfrak{a}}$. Then $\mathcal{O}$ is such that

(1) there exists $w \in W$ such that $w\chi = \frac{1}{2}\check{h} + \nu$ with $\nu \in \mathfrak{z}(\check{e}, \check{h}, \check{f})$,
(2) if χ satisfies property (1) for any other $\check{\mathcal{O}}'$, then $\check{\mathcal{O}}' \subset \overline{\check{\mathcal{O}}}$.

The results in [BM] guarantee that for any χ there is a unique $\check{\mathcal{O}}(\chi)$ satisfying (1) and (2) above. Here is another characterization of the orbit $\check{\mathcal{O}}$. Let

$$\check{\mathfrak{g}}_1 := \{\ x \in \check{\mathfrak{g}}\ :\ [\chi, x] = x\ \}, \qquad \check{\mathfrak{g}}_0 := \{x \in \check{\mathfrak{g}}\ :\ [\chi, x] = 0\ \}.$$

Then $\check{\mathfrak{g}}_0$ has an open dense orbit in $\check{\mathfrak{g}}_1$. The corresponding nilpotent orbit in $\check{\mathfrak{g}}$ is $\check{\mathcal{O}}$.

The pair $(\check{\mathcal{O}}, \nu)$ has further nice properties. For example if $\nu = 0$, then the representation $L(\chi)$ is one of the parameters that the Arthur conjectures predicts to play a role in the residual spectrum. In particular, $L(\chi)$ should be unitary. In the p-adic case, its Iwahori-Matsumoto dual is tempered, and therefore unitary ([BM]). In the real case, a proof of its unitarity is given in [B3], and in section 9 of this paper.

The centralizer $\mathfrak{z}(\check{e}, \check{h}, \check{f})$ is a product of symplectic and orthogonal Lie algebras. We will often abbreviate it as $\mathfrak{z}(\check{\mathcal{O}})$. The orbit $\check{\mathcal{O}}$ is called *distinguished* if $\mathfrak{z}(\check{\mathcal{O}})$ does not contain a nontrivial torus; equivalently, the orbit does not meet any proper Levi component. Assume that $\check{\mathcal{O}}$ is not distinguished. Let $\check{\mathfrak{m}}$ be the centralizer of a Cartan subalgebra in $\mathfrak{z}(\check{\mathcal{O}})$. This is the Levi component of a parabolic subalgebra. Let $M \subset G$ be the Levi whose Lie algebra $\mathfrak{m}$ has $\check{\mathfrak{m}}$ as its dual. The parameter χ gives rise to a spherical irreducible representation $L_M(\chi)$ on M as well as a $L(\chi)$. Then $L(\chi)$ is the unique spherical subquotient of

$$I_M(\chi) := Ind_M^G[L_M(\chi)]. \tag{1.0.2}$$

In the p-adic case (*e.g.* [BM]) there are precise conditions for when $I_M(\chi) = L(\chi)$. In the real case the equality does not hold, but we show that it does for the mulitplicities of the *relevant* K-types (section 4.2).

We will use the data $(\check{\mathcal{O}},\nu)$ to parametrize the unitary dual. Fix an $\check{\mathcal{O}}$. A unitary representation $L(\chi)$ will be called a *complementary series attached to* $\check{\mathcal{O}}$, if the data associated to χ is $\check{\mathcal{O}}$. To describe it, we need to give the set of ν such that $L(\chi)$ with $\chi=\frac{1}{2}\check{h}+\nu$ is unitary. Viewed as an element of $\mathfrak{z}(\check{\mathcal{O}})$, the element ν gives rise to a spherical parameter $((0),\nu)$ where (0) denotes the trivial nilpotent orbit. The main result in section 3.2 says that the ν giving rise to the complementary series for $\check{\mathcal{O}}$ coincide with the ones giving rise to the complementary series for (0) on $\mathfrak{z}(\check{\mathcal{O}})$. This is suggestive of Langlands functoriality.

It is natural to conjecture that such a result will hold for all split groups. Recent work of D. Ciubotaru for F_4, and by D. Ciubotaru and myself for the other exceptional cases, show that there are exceptions.

I give a more detailed outline of the paper. Section 2 reviews notation from earlier papers. Section 3 gives a statement of the main results. The unitarity of the unipotent representations is dealt with in section 8. For the p-adic case I simply cite [BM3]. The real case (sketched in [B2]) is redone in section 9.5. The proofs are simpler than the original ones because I take advantage of the fact that wave front sets, asymptotic supports and associated varieties "coincide" due to [SV]. Section 10.1 proves an irreducibility result which is clear in the p-adic case from the work of Kazhdan-Lusztig. This is needed for determining the complementary series (condition (C3) in section 3.1).

Sections 4 and 5 deal with the nonunitarity. The decomposition $\chi=\frac{1}{2}\check{h}+\nu$ is introduced in section 3. It is more common to parametrize the χ by representatives in $\check{\mathfrak{a}}$ which are dominant with respect to some positive root system. It is quite messy to determine the data $(\check{\mathcal{O}},\nu)$ from a dominant parameter, because of the nature of the nilpotent orbits and the Weyl group. Sections 2.3 and 2.4 give a combinatorial description of $(\check{\mathcal{O}},\nu)$ starting from a dominant χ.

In the classical cases, the orbit $\check{\mathcal{O}}$ is given in terms of partitions. To such a partition we associate a Levi component

$$\check{\mathfrak{m}}:=\check{\mathfrak{g}}_0\times gl(k_1)\times\cdots\times gl(k_r)$$

of a parabolic subalgebra. The intersection of $\check{\mathcal{O}}$ with $\check{\mathfrak{m}}$ is an orbit of the form

$$\check{\mathcal{O}}_0\times(k_1)\times\cdots\times(k_r)$$

where $\check{\mathcal{O}}_0$ is an even nilpotent and (k_i) is the principal nilpotent orbit on $gl(k_i)$. Then χ gives rise to irreducible spherical modules $L_M(\chi)$, $L(\chi)$ and $I_M(\chi)$ as in (1.0.2). The module $L(\chi)$ is the spherical subquotient of $I_M(\chi)$, but the two are not equal. However the multiplicities of the *relevant K-types* are the same. These are representations of the Weyl group in the p-adic case, representations of the maximal compact subgroup in the real case. Their definition is in section 4.2; they are a small finite set of representations which provide necessary conditions for unitarity which are also sufficient.

The relationship between the real and p-adic case is investigated in section 4, in particular the issue is addressed of how the relevant K-types allow us to deal with the p-adic case only.

The determination of the nonunitary parameters proceeds by induction on the rank of $\mathfrak{g}$ and by the inclusion relations of the closure of the orbit $\check{\mathcal{O}}$. Section 5 completes the induction step; it shows that conditions (B) in section 3.1 is necessary. The last part of the induction step is actually done in section 3.1.

I would like to thank David Vogan for generously sharing his ideas about the relation between K-types, Weyl group representations and signatures. They were the catalyst for this paper.

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2. Description of the spherical parameters

2.1. **Explicit Langlands parameters.** We consider spherical irreducible representations of the split connected classical groups of rank n of type $B,\ C,\ D$, precisely, $G = So(2n+1),\ G = Sp(2n)$ and $G = So(2n)$. For spherical representations of split groups, the infinitesimal character determines the Langlands parameter uniquely, so we use the terms interchangeably. As mentioned in the introduction, the infinitesimal character is assumed real ([BM2]) throughout the paper. In the case of classical groups it can be represented by a vector of size the rank of the group. Two such vectors represent the same infinitesimal charater if they are conjugate via the Weyl group which acts by permutations and sign changes for type $B,\ C$ and by permutations and an even number of sign changes in type D. For a given infinitesimal character χ, let $L(\chi)$ be the corresponding irreducible module.

For any nilpotent orbit $\check{\mathcal{O}}$ we attach a parameter $\chi_{\check{\mathcal{O}}}$ as follows. Let $\{\check{e}, \check{h}, \check{f}\}$ be representatives for the Lie triple associated to a nilpotent orbit $\check{\mathcal{O}}$. Then $\chi_{\check{\mathcal{O}}} := \frac{1}{2}\check{h}$.

To each χ we will attach a nilpotent orbit $\check{\mathcal{O}} \subset \check{\mathfrak{g}}(n)$ and a parabolic subgroup with Levi component $M(\chi) := G_0(n_0) \times GL(k_1) \times \cdots \times GL(k_r)$. In addition we will attach to χ an even nilpotent orbit $\check{\mathcal{O}}_0 \subset \check{\mathfrak{g}}_0(m)$ and spherical characters $\chi_0 := \chi_{\check{\mathcal{O}}_0}$ and χ_i on $GL(k_i)$ such that $L(\chi)$ is the spherical subquotient of

$$Ind_{M(\chi)}^{G}[L(\chi_0) \otimes \chi_i]. \tag{2.1.1}$$

We omit the unipotent radical of P from the notation since it is not so useful for our purpuses.

The induced module and the irreducible spherical quotient are very close to being equal. The multiplicities of certain *relevant* K-types defined in section 4 are the same for $L(\chi)$ and the induced module (2.1.1).

2.2. We introduce the following notation (a variant of the one used by Zelevinski [ZE]).

Definition. *A* string *is a sequence*

$$(a, a+1, \dots, b-1, b)$$

of numbers increasing by 1 from a to b. A set of strings is called nested *if the entries of any two such strings, say* $(a_1, \dots, b_1)$ *and* $(a_2, \dots, b_2)$*, differ by integers and either*

$$a_1 \le a_2 \le b_2 \le b_1 \qquad \text{or} \qquad a_2 \le a_1 \le b_1 \le b_2,$$

or else

$$b_1 + 1 < a_2 \qquad \text{or} \qquad b_2 + 1 < a_1. \quad \square$$

Each string represents a 1-dimensional spherical representation of a $GL(n_i)$ with $n_i = b_i - a_i + 1$. A set of strings represents an induced module from the corresponding representation $\otimes\chi_i$ on the Levi component $M = \prod GL(n_i)$. If the strings are nested, there is no way of combining the entries of any two such strings to form a strictly longer one. This property has to do with irreducibility; the induced module to $GL(n_1 + \dots)$ is irreducible and the resulting module is independent of the order of the strings or equivalently the factors $GL(n_i)$.

On the other hand, consider a vector of size n interpreted as an infinitesimal character χ of some $GL(n)$. Then there is only one way to make a nested set of strings. Separate the entries into subsets $A_1, \dots, A_k$ so that the entries in each A_i differ by integers, but they do not for entries in A_i, A_j for $i \neq j$. Order the entries in A_1 in increasing order. Then extract the longest sequence starting with the smallest element in A_1 that can form a string. Continue to extract sequences from the remainder until there are no elements left. Apply the procedure to the other A_i.

2.3. **Nilpotent orbits.** In this section we attach a set of parameters to each nilpotent orbit $\check{\mathcal{O}}$. Let $\check{e}, \check{h}, \check{f}$ be a Lie triple so that $\check{e} \in \check{\mathcal{O}}$, and let $\mathfrak{z}(\check{\mathcal{O}})$ be its centralizer. The parameters χ attached to $\check{\mathcal{O}}$ are of the form

$$\chi = \frac{1}{2}\check{h} + \nu, \qquad \nu \in \mathfrak{z}(\check{\mathcal{O}}), \quad \text{semisimple.} \tag{2.3.1}$$

The parameters χ satisfy the condition that if

$$\chi = \frac{1}{2}\check{h}' + \nu', \qquad \nu' \in \mathfrak{z}(\check{\mathcal{O}}'), \quad \text{semisimple,} \tag{2.3.2}$$

then $\check{\mathcal{O}}' \subset \overline{\check{\mathcal{O}}}$. By [BM], given χ, then $\check{\mathcal{O}}$ exists and is unique.

Type B. The orbit $\check{\mathcal{O}} \subset sp(2n, \mathbb{C})$, so it is parametrized by a partition

$$(\underbrace{1,\dots,1}_{r_1},\underbrace{2,\dots,2}_{r_2},\dots,\underbrace{j,\dots,j}_{r_j},\dots). \tag{2.3.3}$$

with r_{2i-1} even. We write $\frac{1}{2}\check{h}$ in the usual coordinates as

$$(\underbrace{0,\dots,0}_{s_0},\underbrace{1/2,\dots,1/2}_{s_{1/2}},\dots,\underbrace{j,\dots,j}_{s_j},\dots) \tag{2.3.4}$$

where

$$s_0 = (\sum r_{2i-1})/2,\ s_{k-1/2} = \sum_{i>k} r_{2i},\ s_k = \sum_{i>k} r_{2i-1}, \quad k \geq 1. \tag{2.3.5}$$

If one of the $r_i > 1$, then the nilpotent orbit $\check{\mathcal{O}}$ meets a Levi component $\check{\mathfrak{g}}(n-i) \times gl(i)$ and the intersection is an orbit $\check{\mathcal{O}}(n-i) \times (i)$, where (i) denotes the principal nilpotent orbit in $gl(i)$. We can extract the string $(-\frac{i-1}{2},\dots,\frac{i-1}{2})$, and rewrite the parameter (2.3.4) as

$$(\underbrace{0,\dots,0}_{s'_0},\underbrace{1/2,\dots,1/2}_{s'_{1/2}},\dots,\underbrace{j,\dots,j}_{s'_j},\dots;-\frac{i-1}{2},\dots,\frac{i-1}{2}). \tag{2.3.6}$$

We get

$$\begin{aligned} s'_0 &= \begin{cases} s_0 - 1 & \text{if i is odd,} \\ s_0 & \text{if i is even,} \end{cases} \\ s'_j &= \begin{cases} s_j - 2 & \text{if } j \leq (i-1)/2, i \equiv j-1 (mod\ 2), \\ s_j & \text{otherwise.} \end{cases} \end{aligned} \tag{2.3.7}$$

Rewrite the partition as $(2x_0,\dots,2x_{2m},a_1,a_1,\dots a_k,a_k)$ with $x_i < x_{i+1}$ and possibly $x_0 = 0$ to make the number of x_i odd. A general parameter χ attached to $\check{\mathcal{O}}$ can be rewritten as

$$(\underbrace{1/2,\dots,1/2}_{m},\dots,x_{2m}-1/2;\dots;-\frac{a_i-1}{2}+\nu_i,\dots,\frac{a_i-1}{2}+\nu_i;\dots). \tag{2.3.8}$$

Condition (2) in the introduction, which says that χ cannot be written as $\frac{1}{2}\check{h}' + \nu$ for any larger nilpotent $\check{\mathcal{O}}'$, translates into the following:

(1) the strings with $\frac{a_i-1}{2} + \nu_i - \frac{a_j-1}{2} - \nu_j \in \mathbb{Z}$ are nested.
(2) the strings with $\frac{a_i-1}{2} + \nu_i - 1/2 \in \mathbb{Z}$ satisfy the additional condition that either $x_{2m} + 1/2 < -\frac{a_i-1}{2} + \nu_i$ or there is j such that

$$x_j + 1/2 < -\frac{a_i-1}{2} + \nu_i \leq \frac{a_i-1}{2} + \nu_i < x_{j+1} + 1/2. \tag{2.3.9}$$

We lump the strings $(-\frac{a_1-1}{2},\dots,\frac{a_i-1}{2})$ for a_i even together with the parameter coming from the x_i with $x_i < x_{i+1}$, and rewrite χ in the form (2.3.8).

The first part, denoted χ_0, is as in (2.3.4), and corresponds to an even nilpotent orbit $\check{\mathcal{O}}_0 \subset \check{\mathfrak{m}}(n_0)$ with partition $(2x_0, \dots, 2x_{2m})$ satisfying $x_i \le x_{i+1}$, and will be called *unipotent*. We will rewrite the remaining strings as

$$\chi_i := (f_i + \nu_i, \dots, F_i + \nu_i) \qquad f_i, F_i \equiv \frac{1}{2}(\mathbb{Z}), \quad 0 \le \nu_i \le 1/2 \tag{2.3.10}$$

This presentation of the strings is unique except when $\nu_j = 1/2$. The reason is as follows. Any string can be written as

$$(f + \nu, \dots, F + \nu) \tag{2.3.11}$$

with $f, F \equiv 1/2(\mathbb{Z})$ and $0 < \nu < 1$. If $\nu > 1/2$, then replace it by $\nu = 1 - \tilde{\nu}$ and reverse the signs and order to get

$$(-F - 1 + \widetilde{\nu}, \dots, -f - 1 + \widetilde{\nu}), \tag{2.3.12}$$

which is of the form (2.3.10). In case $\nu_j = 1/2$, we can represent each string as either $(f+1/2, \dots, F+1/2)$ or $(-F-1+1/2, \dots, -f-1+1/2)$. We choose the expression with the leftmost number being larger in absolute value.

The setup attaches to each parameter a Levi component $\check{\mathfrak{m}} := \check{\mathfrak{g}}(n_0) \times gl(n_1) \times \dots \times gl(n_k)$ such that $L(\chi)$ is the spherical subquotient of the induced module

$$Ind_M^G[L(\chi_0) \otimes \chi_1 \otimes \dots \otimes \chi_k] \tag{2.3.13}$$

Type C. The orbit $\check{\mathcal{O}} \subset so(2n+1, \mathbb{C})$, so it is parametrized by a partition

$$(\underbrace{1, \dots, 1}_{r_1}, \underbrace{2, \dots, 2}_{r_2}, \dots, \underbrace{j, \dots, j}_{r_j}, \dots). \tag{2.3.14}$$

with r_{2i+1} even. We write $\frac{1}{2}\check{h}$ is in the usual coordinates as

$$(\underbrace{0, \dots, 0}_{s_0}, \underbrace{1/2, \dots, 1/2}_{s_{1/2}}, \dots, \underbrace{j, \dots, j}_{s_j}, \dots) \tag{2.3.15}$$

where

$$s_0 = (\sum r_{2i-1} - 1)/2, \ s_{k-1/2} = \sum_{i>k} r_{2i}, \ s_k = \sum_{i>k} r_{2i-1}, \quad k \ge 1. \tag{2.3.16}$$

If one of the $r_i > 1$, then the nilpotent orbit $\check{\mathcal{O}}$ meets a Levi component $\check{\mathfrak{g}}(n-i) \times gl(i)$ and the intersection is an orbit $\check{\mathcal{O}}(n-i) \times (i)$, where (i) denotes the principal nilpotent orbit in $gl(i)$. We can extract a string $(-\frac{i-1}{2}, \dots, \frac{i-1}{2})$, and rewrite the parameter (2.3.15)

$$(\underbrace{0, \dots, 0}_{s_0'}, \underbrace{1/2, \dots, 1/2}_{s_{1/2}'}, \dots, \underbrace{j, \dots, j}_{s_j'}, \dots; -\frac{i-1}{2}, \dots, \frac{i-1}{2}). \tag{2.3.17}$$

We get as in (2.3.7),

$$s_0' = \begin{cases} s_0 - 1 & \text{if i is odd,} \\ s_0 & \text{if i is even,} \end{cases}$$

$$s_j' = \begin{cases} s_j - 2 & \text{if } j \le (i-1)/2, i \equiv j-1 (mod\ 2), \\ s_j & \text{otherwise.} \end{cases}$$

Rewrite the partition as $(2x_0+1, \dots, 2x_{2m}+1, a_1, a_1, \dots a_k, a_k)$ with $x_i < x_{i+1}$. A general parameter χ attached to $\check{\mathcal{O}}$ can be rewritten as

$$(\underbrace{0, \dots, 0}_{m}, \dots, x_{2m}; \dots ; -\frac{a_i - 1}{2} + \nu_i, \dots, \frac{a_i - 1}{2} + \nu_i; \dots). \tag{2.3.18}$$

Condition (2) in the introduction, which says that χ cannot be written as $\frac{1}{2}\check{h} + \nu$ for a larger nilpotent $\check{\mathcal{O}}'$, translates into the following:

(1) the strings with $\frac{a_i-1}{2} + \nu_i - \frac{a_j-1}{2} - \nu_j \in \mathbb{Z}$ are nested,
(2) the strings with $\frac{a_i-1}{2} + \nu_i \in \mathbb{Z}$ satisfy the additional condition that either $x_{2m} + 1 < -\frac{a_i-1}{2} + \nu_i$ or there is j such that

$$x_j + 1 < -\frac{a_i - 1}{2} + \nu_i \le \frac{a_i - 1}{2} + \nu_i < x_{j+1} + 1. \tag{2.3.19}$$

We lump the strings $(-\frac{a_i-1}{2}, \dots, \frac{a_i-1}{2})$ with a_i odd together with the parameter coming from the x_i with $x_i < x_{i+1}$, and rewrite χ in the form (2.3.18). The first part, denoted χ_0 is as in (2.3.16), and corresponds to an even nilpotent orbit $\check{\mathcal{O}}_0 \subset \check{\mathfrak{m}}(n_0)$ with partition $(2x_0+1, \dots, 2x_{2m})$ with $x_i \le x_{i+1}$, and will be called *unipotent*.

We will rewrite the remaining strings as

$$\chi_i := (f_i + \nu_i, \dots, F_i + \nu_i) \qquad f_i, F_i \equiv 0(\mathbb{Z}), \quad 0 \le \nu_i \le 1/2 \tag{2.3.20}$$

This presentation of the strings is unique except when $\nu_j = 1/2$ for the same reason as in type B. We make the same choice as in that case.

The setup attaches to each parameter a Levi component $\check{\mathfrak{m}} := \check{\mathfrak{g}}(n_0) \times gl(n_1) \times \dots \times gl(n_k)$ such that $L(\chi)$ is the spherical subquotient of the induced module

$$Ind_M^G[L(\chi_0) \otimes \chi_1 \otimes \dots \otimes \chi_k]. \tag{2.3.21}$$

Type D. The orbit $\check{\mathcal{O}} \subset so(2n, \mathbb{C})$, so it is parametrized by a partition

$$(\underbrace{1, \dots, 1}_{r_1}, \underbrace{2, \dots, 2}_{r_2}, \dots, \underbrace{j, \dots, j}_{r_j}, \dots). \tag{2.3.22}$$

with r_{2i+1} even. We write $\frac{1}{2}\check{h}$ in the usual coordinates as

$$(\underbrace{0, \dots, 0}_{s_0}, \underbrace{1/2, \dots, 1/2}_{s_{1/2}}, \dots, \underbrace{j, \dots, j}_{s_j}, \dots) \tag{2.3.23}$$

where

$$s_0 = (\sum r_{2i-1})/2,\ s_{k-1/2} = \sum_{i>k} r_{2i},\ s_k = \sum_{i>k} r_{2i-1}, \quad k \geq 1. \tag{2.3.24}$$

If one of the $r_i > 1$, then the nilpotent orbit $\check{\mathcal{O}}$ meets a Levi component $\check{\mathfrak{g}}(n-i) \times gl(i)$ and the intersection is an orbit $\check{\mathcal{O}}(n-i) \times (i)$, where (i) denotes the principal nilpotent orbit in $gl(i)$. We rewrite the parameter (2.3.23)

$$(\underbrace{0,\dots,0}_{s_0'},\underbrace{1/2,\dots,1/2}_{s_{1/2}'},\dots,\underbrace{j,\dots,j}_{s_j'},\dots;-\frac{i-1}{2},\dots,\frac{i-1}{2}) \tag{2.3.25}$$

where s_j' are as in (2.3.7).

Rewrite the partition as $(2x_0+1,\dots,2x_{2m-1}+1,a_1,a_1,\dots a_k,a_k)$ with $x_i < x_{i+1}$. A general parameter χ attached to $\check{\mathcal{O}}$ can be rewritten as

$$(\underbrace{0,\dots,0}_{m},\dots,x_{2m-1}-1;\dots;-\frac{a_i-1}{2}+\nu_i,\dots,\frac{a_i-1}{2}+\nu_i;\dots). \tag{2.3.26}$$

Condition (2) in the introduction, which says that χ cannot be written as $\frac{1}{2}\check{h}+\nu$ for a larger nilpotent $\check{\mathcal{O}}'$, translates into the following:

1. the strings with $\frac{a_i-1}{2}+\nu_i-\frac{a_j-1}{2}-\nu_j \in \mathbb{Z}$ are nested,
2. the strings with $\frac{a_i-1}{2}+\nu_i \in \mathbb{Z}$ satisfy the additional condition that either $x_{2m}+1 < -\frac{a_i-1}{2}+\nu_i$ or there is j such that

$$x_j+1 < -\frac{a_i-1}{2}+\nu_i \leq \frac{a_i-1}{2}+\nu_i < x_{j+1}+1. \tag{2.3.27}$$

We lump the strings $(-\frac{a_i-1}{2},\dots,\frac{a_i-1}{2})$ with a_i odd together with the parameter coming from the x_i with $x_i < x_{i+1}$, and rewrite χ in the form (2.3.26). The first part, denoted χ_0, is as in (2.3.24), and corresponds to an even nilpotent orbit $\check{\mathcal{O}}_0 \subset \check{\mathfrak{m}}(n_0)$ with partition $2x_0+1,\dots 2x_{2m-1})$ with $x_i \leq x_{i+1}$, and will be called *unipotent*.

We will rewrite the remaining strings as

$$\chi_i := (f_i+\nu_i,\dots,F_i+\nu_i) \qquad f_i, F_i \equiv 0(\mathbb{Z}), \quad 0 \leq \nu_i \leq 1/2 \tag{2.3.28}$$

This presentation of the strings is unique except when $\nu_j = 1/2$. We adopt the same conventions as in types B, C.

The setup attaches to each parameter a Levi component $\check{\mathfrak{m}} := \check{\mathfrak{g}}(n_0) \times gl(n_1)\times\dots\times gl(n_k)$ such that $L(\chi)$ is the spherical subquotient of the induced module

$$Ind_M^G[L(\chi_0)\otimes\chi_1\otimes\dots\otimes\chi_k] \tag{2.3.29}$$

Remark. In type D, a (real) spherical parameter λ is hermitian if and only if there is $w \in W(D_n)$ such that $w\lambda = -\lambda$. If the parameter has entries equal to zero, then the analysis above is sufficient. If not, there are two inequivalent spherical parameters, one for λ and another for λ' obtained by applying the outer automorphism of order two. They are either both unitary or both nonunitary. With the description above, one string might have to be written as

$$(-f-\nu,-f+1+\nu,\dots,F+\nu). \tag{2.3.30}$$

If so, we can always consider the other parameter related to it by the outer automorphism.

2.4. **Relation between infinitesimal characters and strings.** We also need to show how to obtain this parametrization in terms of strings from the infinitesimal character.

Type B. Partition the coordinates of χ into subsets $A_1, A_2, \dots$ according to the relation

$$\nu_i \sim \nu_j \text{ if and only if } \nu_i+\nu_j \text{ or } \nu_i-\nu_j \in \mathbb{Z}.$$

Write the part formed of half-integers in increasing order

$$(r-1/2,\dots,r-1/2,\dots,R-1/2,\dots,R-1/2),\quad 0<r\le R \text{ integers.} \tag{2.4.1}$$

Then extract the longest possible string $(r-1/2,\dots,l-1/2)$. The remaining parameter is of the same form, so we can continue extracting strings until there are no coordinates left. The strings that start with $1/2$ are lumped together to form the *spherical unipotent representation* part of the parameter. For example if the parameter is

$$(1/2,1/2,1/2,3/2,3/2,3/2,5/2,5/2,5/2,5/2,7/2),$$

then the strings are

$$(1/2,3/2,5/2,7/2),\ (1/2,3/2,5/2),\ (1/2,3/2,5/2),\ (5/2),$$

and the parameter is

$$(1/2,1/2,3/2,5/2,5/2,7/2\ ;\ 5/2)$$

corresponding to the nilpotent orbit $\check{\mathcal{O}}=(8,6,6,1,1)$ with $\check{\mathcal{O}}_0=(8,6,6)$. For the set formed of integers, write the coordinates in increasing order as

$$(r,\dots,r,\dots,R,\dots,R),\ 0\le r\le R \text{ integers.} \tag{2.4.2}$$

Extract the longest possible string $(l,\dots,0,\dots,k)$ (or just $(l,\dots,k)$ with $l>0$ if there are no entries equal to 0) by changing entries to their negatives if necessary. Assume as we may that $k\ge |l|$. The remainder is of the same form as in (2.4.2), so we can continue until there are no entries left. Then rewrite the strings in increasing order as in section 2.3.

For example, if the parameter is

$$(0,0,1,1,1,1,2,3,3,4,5),$$

the strings are, following the conventions of section 2.3,

$$(-5,-4,-3,-2,-1,0,1),\ (-1,0,1),\ (-4,-3). \tag{2.4.3}$$

This adds the pairs $(7,7,3,3,2,2)$ to the partition corresponding to $\check{\mathcal{O}}_0$ to form $\check{\mathcal{O}}$.

For a part which is neither integers nor half-integers, change signs if necessary and rearrange in increasing order

$$(r+\nu,\dots,r+\nu,\dots,R+\nu,\dots,R+\nu),\ 0<\nu<1/2,\quad r\le R \text{ half-integers.} \tag{2.4.4}$$

Then extract the longest possible string $(r+\nu,\dots,l+\nu)$. The remainder is of the same type, so apply the same procedure until there are no entries left.

For example, if the parameter is

$$(1/4,1/4,3/4,5/4,5/4),$$

rewrite it as

$$(-5/4,-5/4,-1/4,-1/4,3/4),$$

and then extract the strings

$$(-5/4,-1/4,3/4),\ (-5/4,-1/4).$$

Type C. Partition the coordinates as for type B. Write the integer part in increasing order

$$(r,\dots,r,\dots,R,\dots,R),\qquad 0\le r\le R. \tag{2.4.5}$$

Extract the longest possible string $(l,\dots,0,\dots,k)$ with the same conventions as for type B right after (2.4.2). The remaining parameter is of the same form, so we can continue until there are no zeroes left. The number of strings is the m for the *special unipotent* part of the parameter, and furthermore, $k=x_{2m},\ l=x_{2m-1}$. After that, extract the longest possible strings of the form $(l,\dots,k)$ with $k\ge l>0$ until there are no entries left. If the longest such string starts with a 1, then its largest entry is x_0, otherwise set $x_0=0$. For example, if the strings are as in (2.4.3), the unipotent parameter has $\check{\mathcal{O}}_0=(11,3,3,3,1)$, $\check{\mathcal{O}}=(11,3,3,3,2,2,1)$, and the unipotent part of the parameter is

$$(0,0,1,1,1,1,2,3,4,5).$$

For the half-integer part, write it as

$$(r-1/2,\dots,r-1/2,\dots,R-1/2,\dots,R-1/2),\qquad 0\le r\le R \text{ integers.} \tag{2.4.6}$$

Change signs of coordinates if necessary, and extract the longest possible string of the form $(l-1/2,\dots,k+1/2)$. The remainder is of the same form as in (2.4.6), so we can repeat the procedure until there are no entries left. Rewrite the strings to conform to the conventions of section 2.3.

For the rest of the parameter which is formed of neither integers nor half-integers, use the same method as for type B.

Type D. Partition the coordinates as for types B, C. Write the integer part in increasing order

$$(r,\dots,r,\dots,R,\dots,R),\qquad 0\le r\le R. \tag{2.4.7}$$

Extract the longest possible string $(l, \dots, 0, \dots, k)$ with the same conventions as for type B right after (2.4.2). The remaining parameter is of the same form, so repeat the procedure until there are no zeroes left. The number of strings is the m for the *special unipotent* part of the parameter. Furthermore, assuming as we may that $|l| \le k$, we get $k = x_{2m},\ l = x_{2m-1}$. After that, extract the longest possible strings of the form $(l, \dots, k)$ with $k \ge l > 0$ until there are no entries left. For example, for the strings in (2.4.3, the nilpotent orbit $\check{\mathcal{O}}_) = (11, 3, 3, 3)$ and $\check{\mathcal{O}} = (11, 3, 3, 3, 2, 2)$.

For the half-integer part, write it as

$$(r - 1/2, \dots, r - 1/2, \dots, R - 1/2, \dots, R - 1/2), \qquad 0 \le r \le R. \quad (2.4.8)$$

Then extract the longest possible string of the form $(l - 1/2, \dots, k + 1/2)$, by changing entries into their negatives if necessary. The remainder is of the same form as in (2.4.8), so we can continue until there are no entries left. Rewrite the strings as in section 2.3.
For the rest of the parameter, which is formed of neither integers nor half-integers, apply the same procedure as for types B, C.

Note also that the Weyl group of type D consists of permutations and even number of sign changes. if we change a single sign, we are applying an outer automorphism. but the remark from section 2.3 applies.

2.5. **Summary.** In conclusion, to each spherical parameter χ we have associated a nilpotent orbit $\check{\mathcal{O}}$ in the dual algebra and a Levi component $M = G_0 \times GL(k_1) \times \dots \times GL(k_r)$. The nilpotent orbit $\check{\mathcal{O}}$ meets the dual algebra $\check{\mathfrak{g}}_0 \times gl(k_1) \times \dots gl(k_r)$ to give an even nilpotent $\check{\mathcal{O}}_0$ on $\check{\mathfrak{g}}_0$ and the principal nilpotent orbit on each of the $gl(k_i)$. In addition we have associated a spherical parameter $(\chi_0, \chi_1, \dots, \chi_r)$ such that χ_0 is half the semisimple element of the Lie triple corresponding to $\check{\mathcal{O}}_0$, and the other χ_i are 1-dimensional. Then $L(\chi)$ is the spherical subquotient of

$$Ind_M^G[L(\chi_0) \otimes \chi_1 \otimes \dots \otimes \chi_r]. \quad (2.5.1)$$

2.6.

Theorem. *The module $I_M(\chi)$ equals $L(\chi)$ in the p-adic case. In the real case equality holds whenever the coordinates of the χ_i with $i \ge 1$ are not congruent to the coordinates of χ_0 modulo $\mathbb{Z}$. In the real case in general, the multiplicities of the* relevant *K-types in $L(\chi)$ and $I_M(\chi)$ coincide.*

Proof. In the p-adic case it follows from [BM] that the two modules are equal. In the real case, if the ν_j in the character χ_j is not an integer or half-integer and $M := G(n - n_j) \times GL(n_j)$ and $L_M := L(\chi_0, \dots, \widehat{\chi_j}, \dots, \chi_r)$ then

$$I_M := Ind_M^G[L_M \otimes \chi_j] \quad (2.6.1)$$

is irreducible by the Kazhdan-Lusztig conjectures for nonintegral infinitesimal character. We omit the details, see [ABV]. The last assertion is a rewording of theorem 5.3 and corollary 5.3. $\square$

2.7. We record the following refinement of theorem 2.6. Let $\check{\mathcal{O}}$ be a nilpotent orbit corresponding to the partition

$$(\underbrace{x_0,\dots,x_0}_{r_0},\dots,\underbrace{x_m,\dots,x_m}_{r_m}). \tag{2.7.1}$$

Assume that $r_i>1$ for some i, and denote x_i by x. Then $\check{\mathcal{O}}$ meets the parabolic subalgebra $\check{\mathfrak{m}}=\check{\mathfrak{g}}(n-x)\times gl(x)$. Let $\check{\mathcal{O}}_M$ be the nilpotent orbit in $\check{\mathfrak{g}}(n-x)$ corresponding to the partition

$$(\underbrace{x_0,\dots,x_0}_{r_0},\dots,\underbrace{x_i,\dots,x_i}_{r_i-2},\dots,\underbrace{x_m,\dots,x_m}_{r_m}).$$

Write $\chi:=\chi_{\check{\mathcal{O}}},\ \chi_M:=\chi_{\check{\mathcal{O}}_M}$. Then $L(\chi)$ is the spherical subquotient of $I_M(\chi:=Ind_M^G[L(\chi_M)\otimes triv]$.

Theorem. *The representation $I_M(\chi)$ is irreducible in the following cases:*

1. *Type A, all cases,*
2. *Type B, whenever $r_i>2$ or x is odd,*
3. *Type C,D whenever $r_i>2$ or x is even.*

The representation $I_M(\chi)$ is reducible in all other cases.

Proof. In the p-adic case this is again [BM]. For the real case, the proof is in section 10. □

3. The Main Result

3.1. **First formulation.** The main result can be summarized as follows.

Theorem. *A spherical representation is unitary if and only if it is a complementary series from an induced from a special unipotent representation tensored with a GL–complementary series.*

We now describe these parameters in precise terms. We assume the conventions in section 2. In particular recall ϵ in definition 2.3, $\epsilon=1/2$ for type B and $\epsilon=0$ for types $C,\ D$.

Let $L(\chi)$ be spherical with $\chi=(\chi_0,\chi_1,\dots,\chi_k)$. We will want to deform the ν's that the representation stays induced irreducible, but we will also consider the endpoint of such an interval and the corresponding spherical factor.

The necessary conditions for unitarity are (A), (B), (AB), and (C1)-(C3). For condition (B) we need the following definition.

Definition. *Set $\epsilon=1/2$, in type B and $\epsilon=0$ in type $C,\ D$. We say a string $(f+\nu,\dots,F+\nu)$ with $f,F\equiv\epsilon(mod\ 2)$ is* adapted, *if it is*

of even *length in type B,*
of odd *length in type $C,\ D$.*

Otherwise we say it is not adapted. *Two strings are of the* same type *if they are both adapted or both not adapted.* □

- **(A):** The part of the parameter formed of entries congruent to ϵ $(mod\ \mathbb{Z})$ is *special unipotent* .
- **(B):** Any string that is *not adapted* is of the form

$$(-E+\nu,\dots,E-1+\nu) \qquad 0<\nu\le 1/2, \quad E\equiv\epsilon(\mathbb{Z}). \tag{3.1.1}$$

This string is of size $2E$.

Any *adapted* string is of the form

$$(-E+\nu,\dots,E+\nu) \qquad 0<\nu\le 1/2, \quad E\equiv\epsilon(\mathbb{Z}) \tag{3.1.2}$$

or

$$(-E-1+\nu,\dots,E-1+\nu) \qquad 0<\nu\le 1/2, \quad E\equiv\epsilon(\mathbb{Z}). \tag{3.1.3}$$

These strings are of size $2E+1$.

The reason for the definition of adapted is as follows. Suppose for simplicity that $k=1$ *i.e.* there is just one string χ_1 aside from χ_0. If it is as in (3.1.1), deforming ν_1 to $1/2$ gives a unitarily induced irreducible representation. But if it is as in (3.1.2) or (3.1.3), deforming ν_1 to 0 or 1 respectively gives a unitarily induced module which is not necessarily irreducible (*cf.* (C3) and theorem 2.7).

In general, suppose $L(\chi)=I_M$ with notation as in the proof of theorem 2.6. If we can deform a ν_j to $1/2$ in (3.1.1), 0 in (3.1.2) or 1 in (3.1.3) and no reducibility occurs, then L is unitary if and only if L_M is unitary. So it is enough to decide the unitarizability of L_M on the lower rank group. In such a situation we say that *we can remove* χ_j. The following lemma and proposition are useful in formulating the necessary conditions (AB) and (C1).

Lemma. *The representation corresponding to a pair of strings* $(-E-\nu,\dots E-\nu)$ *and* $(-E+\nu,\dots,E+\nu)$ *in* $GL(n)$ *is a complementary series for* $\nu<\frac{1}{2}$, *and is not unitary for* $\nu>\frac{1}{2}$, $2\nu\notin\mathbb{Z}$.

Proof. This is well known and goes back to [Stein] (*e.g.* [T] and [V1]). □

Definition. *Consider two strings of the same type as in (3.1.1-3.1.3) with parameters* $\nu_1,\ \nu_2$. *We say that they are* adjacent, *if they have the same type and same* E, *and either* $\nu_1=\nu_2$, *or* $\nu_1\ne\nu_2$ *and there is no other string of the same type with the same* E *and parameter* ν *in between* $\nu_1,\ \nu_2$.

Proposition. *Assume that (A) and (B) are satisfied. Deform a* ν_j *in a string as in (3.1.1-3.1.3) in the interval* $(0,1/2]$.

1. *For case (3.1.1) no reducibility can occur.*
2. *For case (3.1.2) reducibility occurs only if the parameter has a string of type (3.1.3) with the same* E.

(3) *For the case (3.1.3) reducibility occurs only if there is a string of type (3.1.2) with the same E.*

Proof. Consider a string as in (3.1.1) with $0 < \nu_j < 1/2$ and deform ν_j; call the deformed parameter ν. If the induced representation becomes reducible, there must be another string $(f + \nu_i, \dots, F + \nu_i)$ such that $F + \nu_i - E - \nu$ or $F + \nu_i + E + \nu$ is an integer. Since $E \pm F \in \mathbb{Z}$, and $0 < \nu,\ \nu_i \le 1/2$, we must have $\nu = \nu_i$. Furthermore,

$$-E - 1 < f \le E < F \qquad \text{or} \qquad f < -E - 1 \le F < E. \tag{3.1.4}$$

Neither case can occur given that the strings are of type (3.1.1-3.1.3).

Deform a string as in (3.1.2). For reducibility to occur there has to be another string $(f + \nu_i, \dots, F + \nu_i)$ such that

$$-E < f \le E < F \qquad \text{or} \qquad f < -E \le F < E. \tag{3.1.5}$$

The first case implies $F + f > 0$ which cannot occur given (3.1.1-3.1.3). In the second case, the choices are

$$-F < -E \le F - 1 < E,\ -F < -E \le F < E,\ -F - 1 < -E \le F - 1 < E.$$

It follows that $F = E$ and the string with ν_i is type (3.1.3). The last case is similar. ☐

The argument above shows that we can remove strings of type (3.1.1) by a complementary series argument deforming ν_j to $1/2$.

Thus assume there are no strings of type (3.1.1).

Suppose there are two adjacent strings with the same E as in (3.1.2) with parameters $\nu_i \le \nu_j$. Then L is induced from an $L_M \otimes \chi_i \otimes \chi_j$. No reducibility occurs when we deform ν_i to ν_j. When $\nu_i = \nu_j$, the representation is unitarily induced irreducible from a $\pi \otimes L_M$ on $GL(4E + 2)G(n - 4E - 2)$ where π is a complementary series as in lemma 3.1. Thus L is unitary if and only if L_M is unitary; we can *remove* $\chi_i,\ \chi_j$ from the parameter. In case (3.1.3) the representation π is *not* unitary and we conclude that L is **not** unitary.

We summarize these properties.

(AB): The strings for a fixed E that are not adapted all come from complementary series from induced from $Triv \otimes L_M$ on $GL(2E)G(n-2E)$. Adjacent strings of same E, and adapted as in (3.1.2), are in a complementary series from an $\pi_j \otimes L_M$ on a $Gl(4E+2)G(n-4E-2)$ as in lemma 3.1.

(C1): No two strings of type (3.1.3) with the same E can be adjacent.

The strings in (AB) can be removed from the parameter; the ensuing parameter is unitary if and only if the original parameter is unitary. We have reduced to the case when for a given E, there is at most one string for each value of ν; and they must alternate between (3.1.2) and (3.1.3) with increasing ν. Suppose there is more than one string present, and label the

$\nu's$ as $0 < \nu_1 < \nu_2 < \cdots < \nu_{m-1} < \nu_m$. Suppose the one with parameter ν_m is of type (3.1.3). Necessarily the string corresponding to ν_{m-1} is of type (3.1.2). Then we can deform ν_m upwards to $1 - \nu_{m-1}$, and see that this is induced irreducible from a representation which is unitarily induced from a Levi component $GL(4E+2)G(n-4E-2)$ (a complementary series on $GL(4E+2)$). Such a pair can be removed from the parameter; the ensuing parameter is unitary if and only if the original parameter is unitary.

On the other hand, suppose the string with parameter ν_m is of type (3.1.2). Deforming ν_m upwards to $1 - \nu_{m-1}$, gives a parameter that is not unitary by lemma 3.1. Thus

(C2): the strings (3.1.2, 3.1.3) for a given E (assuming there is at most one string for each ν,) must alternate between the two types. If there are strings of type (3.1.3), the one with the largest ν is type (3.1.3).

We can remove such pairs of strings, starting with the largest ν_m; the ensuing parameter is unitary if and only if the original parameter is unitary. We are reduced to the case when there is only one string of type (3.1.2,3.1.3). Reducibility of *unipotent parameters* suggests the following.

(C3): In the case of a single string of type (3.1.2) or (3.1.3) of size E, there must be at least one $x_i = E + \epsilon$ in the tempered parameter.

Thus to see whether a parameter is unitary, one checks whether (A)-(C) are satisfied. First we check that (A) and (B) are satisfied. If so, then remove the GL–complementary series in step (AB). Then check for adjacent strings of type (3.1.3) as in (C1). If none are present, remove the complementary series from step (C2). What should result is either a parameter which is tempered or one as in (C3).

Remark. In the case of type D, the condition for a representation to be hermitian is different from types B and C. There is no change to the argument for (AB), (C1) and (C2), because they involve pairs of strings. For (C3), if there is a single string present, then the tempered part of the parameter must be nontrivial, otherwise the parameter is not hermitian. The argument is unchanged otherwise.

3.2. **Second form.** The main result admits the following more invariant description. Let $\check{G}$ be the (complex) dual group and $\check{A} \subset \check{G}$ the maximal torus dual to A. Assuming as we may that the parameter is real, a spherical irreducible representation corresponds to an orbit of an element $\chi \in \check{\mathfrak{a}}$, the Lie algebra of $\check{A}$. In section 2 we attached a nilpotent orbit $\check{\mathcal{O}}$ in $\check{\mathfrak{g}}$ to such a parameter. Let $\check{e}, \check{h}, \check{f}$ be a Lie triple attached to $\check{\mathcal{O}}$, and let $\chi_{\check{\mathcal{O}}} := \frac{1}{2}\check{h}$. By conjugating $\chi_{\check{\mathcal{O}}}$, we can write

$$\chi := \chi_{\check{\mathcal{O}}} + \underline{\nu} \tag{3.2.1}$$

in such a way that $\underline{\nu}$ centralizes the whole Lie triple. This is as follows. The coordinates of χ_0 go into $\chi_{\check{\mathcal{O}}}$; the corresponding coordinates in $\underline{\nu}$ are 0. The coordinates coming from $gl(n_i)$ parameter can be written as

$$(-\frac{n_i-1}{2},\dots,\frac{n_i-1}{2})+\nu_i(1,\dots,1). \tag{3.2.2}$$

The first part goes into $\chi_{\check{\mathcal{O}}}$, the second one into $\underline{\nu}$. The nilpotent orbit $\check{\mathcal{O}}$ determines a partition

$$(a_1,\dots,a_1,\dots,a_k,\dots,a_k), \qquad a_l<a_{l+1} \tag{3.2.3}$$

where r_l is the sum of the number of $n_i=a_l$ and the number of the elements in the partition for $\check{\mathcal{O}}_0$ equal to a_l.

The centralizer $Z_{\check{G}}(\check{e},\check{h},\check{f})$ has Lie algebra $\mathfrak{z}(\check{\mathcal{O}})$ which is a product of $sp(r_l,\mathbb{C})$ or $so(r_l,\mathbb{C})$ $1\le l\le k$ according to the rule

Type B, D: $sp(r_l)$ for a_l even, $so(r_l)$ for a_l odd,

Type C: $sp(r_l)$ for a_l odd, $so(r_l)$ for a_l even.

The parameter $\underline{\nu}$ gives rise to a spherical one for $\mathfrak{z}(\check{\mathcal{O}})$ as follows. For each l, take the ν_i in (3.2.2) for which $\nu_i=a_l$ and a 0 for each term in the partition of $\check{\mathcal{O}}_0$ equal to a_l. The results in section 3.1 can be written in terms of $\underline{\nu}$. The set for which $L(\chi)$ is unitary will be called the *complementary series attached to* $\check{\mathcal{O}}$.

Theorem. *The complementary series attached to $\check{\mathcal{O}}$ coincides with the one attached to the trivial orbit in $\mathfrak{z}(\check{\mathcal{O}})$. These are:*
B: $0\le\nu_1,\dots,\nu_1<\dots<\nu_k,\dots,\nu_k<1/2$.
C, D: $0\le\nu_1,\dots,\nu_1<\dots<\nu_k,\dots,\nu_k\le 1/2<\nu_{k+1}<\dots<\nu_{k+l}<1$
so that $\nu_i+\nu_j\ne 1$ for $i\ne j$ and there are an even number of ν_i such that $1-\nu_{k+1}<\nu_i\le 1/2$ and an odd number of ν_i such that $1-\nu_{k+j+1}<\nu_i<1-\nu_{k+j}$.

The passage to the parameters in 3.1 is to change ν_{k+j} for types $C,\ D$ to $1-\nu_{k+j}$.

4. Relevant K-types

In the real case we will call a K-type (μ,V) *quasi-spherical* if it occurs in the spherical principal series. By Frobenius reciprocity $V^M\ne 0$ and the Weyl group $W(G,A)$ acts on this space.

The representations of $W(A_{n-1})=S_n$ are parametrized by partitions of n, written as $(a_1,\dots,a_k),\ a_i\le a_{i+1}$. The representations of $W(B_n)\cong W(C_n)$ are parametrized as in [L1] by pairs of partitions

$$(a_1,\dots,a_k)\times(b_1,\dots,b_l),$$
$$a_i\le a_{i+1},\quad b_j\le b_{j+1},\quad \sum a_i+\sum b_j=n. \tag{4.0.4}$$

Precisely the representation parametrized by (4.0.4) is as follows. Let $k=\sum a_i,\ l=\sum b_j$. Recall that $W\cong S_n\rtimes\mathbb{Z}_2^n$. Let χ be the character of $\mathbb{Z}_2^n$

which is trivial on the first k $\mathbb{Z}_2$'s, sign on the rest. Its centralizer in S_n is $S_k \times S_l$. Let σ_1 and σ_2 be the representations of $S_k,\ S_l$ corresponding to the partitions (a) and (b). Then σ, the representation parametrized by (4.0.4), is

$$Ind^{W}_{(S_k\times S_l)\times\mathbb{Z}_2^n}\,[(\sigma_1\times\sigma_2)\times\chi].$$

For $W(D_n)$, the representations are parametrized as in (4.0.4) except that $(a)\times(b)$ and $(b)\times(a)$ parametrize the same representation and when $(a) = (b)$, there are two of them $(a)\times(a)_{I,\ II}$. This is because the restriction of $(a)\times(b)$ to $W(D_n)$ is irreducible when $(a)\neq(b)$ and equal to the restriction of $(b)\times(a)$, while the restriction of $(a)\times(a)$ consists of two nonisomorphic irreducible representations $(a)\times(a)_{I,\ II}$. These are usually easy to deal with.

4.1. **Symplectic Groups.** The group is $Sp(n)$ and the maximal compact subgroup is $U(n)$. The K-types of the form

$$(\underbrace{2,\dots,2}_{r},\underbrace{2,\dots,2}_{m},\underbrace{1,\dots,1}_{k},\underbrace{0,\dots,0}_{l},\underbrace{-1,\dots,-1}_{k},\underbrace{-1,\dots,-1}_{2m}). \tag{4.1.1}$$

are all quasi-spherical. The dual K-types are also quasi-spherical, we ignore them because they behave the same way.

Proposition. *The M-fixed vectors of a K-type μ as in (4.1.1) form an irreducible representation of $W(C_n)$ corresponding to the pair of partitions*

$$(m, k+m, k+l+m)\times(r). \tag{4.1.2}$$

Proof. This follows by induction on the rank of $Sp(n)$ using the restriction formula from $U(n)$ to $U(1)\times U(n-1)$ and the restriction formula from S_n to S_{n-1} and its generalization to $W(C_n)$. Here are the details.

Consider the case $n = 1$. There are only two representations, the trivial representation with highest weight (0) and the symmetric square of the standard representation, (2). Write $U(1) = \{e^{i\theta}\}$. Then we can identify M with $\{\pm1\}$ and the normalizer of the split Cartan subalgebra is $\{\alpha\ :\ \alpha^4 = 1\}$. The assertions are clear.

Consider the case $n = 2$. There are four representations of $U(2)$ with highest weights $(2,0)$, $(1,-1)$, $(2,2)$ and $(0,0)$. The first representation is the symmetric square of the standard representation, the second one is the adjoint representation and the third one is the trivial representation. The subgroup $M \subset U(2)$ can be identified with the diagonal subgroup $(\pm1,\pm1)$ inside $U(1)\times U(1)\subset U(2)$. The Weyl group is generated by the elements

$$\begin{bmatrix} i & 0\\ 0 & 1\end{bmatrix},\quad \begin{bmatrix} 1 & 0\\ 0 & i\end{bmatrix},\quad \begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix}. \tag{4.1.3}$$

The space V^M can be read off from the restriction of the representation to $U(1) \times U(1)$ which is

$$\begin{aligned} (2,0) &\longrightarrow (2) \otimes (0) + (1) \otimes (1) + (0) \otimes (2) \\ (1,-1) &\longrightarrow (1) \otimes (-1) + (0) \otimes (0) + (-1) \otimes (1) \\ (2,2) &\longrightarrow (2) \otimes (2) \\ (0,0) &\longrightarrow (0) \otimes (0) \end{aligned} \tag{4.1.4}$$

The claim for the last one is clear. The third one is 1-dimensional so V^M is 1-dimensional; the Weyl group representation is $(0) \times (2)$. The second one has V^M 1-dimensional and the Weyl group representation is $(11) \times (0)$. For the first one, V^M is 2-dimensional and the Weyl group representation is $(1) \times (1)$. These facts can be read off from explicit realizations of the representations.

Assume that the claim is proved for $n-1$. Choose a parabolic subgroup so that its Levi component is $M' = Sp(n-1) \times GL(1)$ and M is contained in it. Let $H = U(n-1) \times U(1)$ be such that $M \subset M' \cap K \subset H$. The restriction rule from K to H is well known. We will only use the cases when $m = 0$ and either $k = 0$ or $r = 0$. We call these *relevant*. Suppose that $m = 0$ and $k = 0$. The cases when $l = 0$ or $r = 0$ are 1-dimensional and are straightforward. So we only consider $l,\ r > 0$. The K-type restricts to

$$(\underbrace{2,\dots,2}_{l},\underbrace{0,\dots,0}_{r-1}) \otimes (0) \tag{4.1.5}$$

$$(\underbrace{2,\dots,2}_{l-1},1,\underbrace{0,\dots,0}_{r-1}) \otimes (1) \tag{4.1.6}$$

$$(\underbrace{0,\dots,0}_{l-1},\underbrace{0,\dots,0}_{r}) \otimes (1) \tag{4.1.7}$$

Only (4.1.5) and (4.1.7) can contribute to V^M and these are the representations

$$[(l-1) \times (r)] \otimes [(1) \times (0)] \tag{4.1.8}$$

$$[(l) \times (r-1)] \otimes [(0) \times (1)] \tag{4.1.9}$$

The Weyl group representation (4.1.8) can only come from

$$[(1, l-1) \times (r)] \tag{4.1.10}$$

$$[(l) \times (r)] \tag{4.1.11}$$

The case (4.1.10) has $(1, l-1) \times (r-1)]$ in its restriction which does not occur in (4.1.8)-(4.1.9). Thus the claim is proved in this case.

Consider the case $m = 0$ and $r = 0$. The case $k = 0$ is 1-dimensional so straightforward. So assume $k > 0$. If $l > 0$ the K-type restricts to

$$(\underbrace{1,\dots,1}_{k},\underbrace{0,\dots,0}_{l},\underbrace{-1,\dots,-1}_{k-1}) \otimes (-1) \tag{4.1.12}$$
$$(\underbrace{1,\dots,1}_{k-1},\underbrace{0,\dots,0}_{l},\underbrace{-1,\dots,-1}_{k}) \otimes (1) \tag{4.1.13}$$
$$(\underbrace{1,\dots,1}_{k-1},\underbrace{0,\dots,0}_{l+1},\underbrace{-1,\dots,-1}_{k-1}) \otimes (0) \tag{4.1.14}$$
$$(\underbrace{1,\dots,1}_{k},\underbrace{0,\dots,0}_{l-1},\underbrace{-1,\dots,-1}_{k}) \otimes (0) \tag{4.1.15}$$

Only (4.1.14) and (4.1.15) contribute to V^M. The Weyl group representations are

$$[(k-1, k+l) \times (0)] \otimes [(1) \times (0)] \tag{4.1.16}$$
$$[(k, k+l-1) \times (0)] \otimes [(1) \times (0)] \tag{4.1.17}$$

The representations (4.1.17) can only come from the restriction of $(1, k, k+l-1) \times (0)$ or $(k, k+l) \times (0)$. If $k > 1$ the first one contains $(1, k-1, k+l-1) \times (0)$ in its restriction which is not in (4.1.16) or (4.1.17). If $k = 1$ then (4.1.17) does not occur in the rstriction. So $(k, k+l) \times (0)$ has to occur as well. But then the restriction is too large. The claim is proved in this case.

Consider the case when none of $k,\ l,\ m,\ r$ are zero.

The K-type in (4.1.1) restricts to

$$(\underbrace{2,\dots,2}_{r+m},\underbrace{1,\dots,1}_{k},\underbrace{0,\dots,0}_{l},\underbrace{-1,\dots,-1}_{k+2m-1}) \otimes (-1) \tag{4.1.18}$$
$$(\underbrace{2,\dots,2}_{r+m},\underbrace{1,\dots,1}_{k},\underbrace{0,\dots,0}_{l-1},\underbrace{-1,\dots,-1}_{k+2m}) \otimes (0) \tag{4.1.19}$$
$$(\underbrace{2,\dots,2}_{r+m},\underbrace{1,\dots,1}_{k-1},\underbrace{0,\dots,0}_{l+1},\underbrace{-1,\dots,-1}_{k+2m-1}) \otimes (0) \tag{4.1.20}$$
$$(\underbrace{2,\dots,2}_{r+m-1},\underbrace{1,\dots,1}_{k+1},\underbrace{0,\dots,0}_{l},\underbrace{-1,\dots,-1}_{k+2m-1}) \otimes (0) \tag{4.1.21}$$
$$(\underbrace{2,\dots,2}_{r+m},\underbrace{1,\dots,1}_{k-1},\underbrace{0,\dots,0}_{l},\underbrace{-1,\dots,-1}_{k+2m}) \otimes (1) \tag{4.1.22}$$
$$(\underbrace{2,\dots,2}_{r+m-1},\underbrace{1,\dots,1}_{k+1},\underbrace{0,\dots,0}_{l-1},\underbrace{-1,\dots,-1}_{k+2m}) \otimes (1) \tag{4.1.23}$$
$$(\underbrace{2,\dots,2}_{r+m-1},\underbrace{1,\dots,1}_{k},\underbrace{0,\dots,0}_{l+1},\underbrace{-1,\dots,-1}_{k+2m-1}) \otimes (1) \tag{4.1.24}$$
$$(\underbrace{2,\dots,2}_{r+m-1},\underbrace{1,\dots,1}_{k},\underbrace{0,\dots,0}_{l},\underbrace{-1,\dots,-1}_{k+2m}) \otimes (2) \tag{4.1.25}$$

Only (4.1.19-4.1.21) and (4.1.25) can contribute to V^M. By the induction hypothesis, the representations of $W(C_{n-1}) \times W(C_1)$ are

$$[(m, k+m, k+m+l-1) \times (r)] \otimes [(1) \otimes (0)] \qquad (4.1.26)$$
$$[[(m, k-1+m, k-1+m+l) \times (r)] \otimes [(1) \otimes (0)] \qquad (4.1.27)$$
$$[(m-1, k1+m-1, k+1+l+m-1) \times (r)] \otimes [(1) \otimes (0)] \qquad (4.1.28)$$
$$[(m, k+m, k+m+l) \times (r-1)] \otimes [(0) \otimes (1)] \qquad (4.1.29)$$

Now consider the representation (4.1.26). It can only occur in the restriction of

$$[(m, k+m, k+m+l) \times (r)] \qquad (4.1.30)$$
$$[(m, k+m+1, k+m+l) \times (r)] \qquad (4.1.31)$$
$$[[(m+1, k+m, k+m+l) \times (r)] \qquad (4.1.32)$$

The restrictions of (4.1.31) or (4.1.32) contains representations that are not in the list (4.1.26)-4.1.29). ☐

4.2. **Orthogonal groups.** Because we are dealing with the spherical case, we can use the orthogonal groups instead of their connected components. We follow Weyl's conventions to parametrize the representations of $O(n)$. Embed $O(a) \subset U(a)$ in the standard way. An irreducible representation of $O(n)$ is parametrized by

$$(a_1, \dots, a_k, 0, \dots, 0; \epsilon), \qquad a_i \ge a_{i+1},\ \epsilon = \pm 1. \qquad (4.2.1)$$

The ϵ is (sometimes) abbreviated as $\pm$. It is the irreducible representation generated by the highest weight vector of the irreducible representation of $U(a)$ with highest weight

$$(a_1, \dots, a_k, \underbrace{1, \dots, 1}_{n-(1-\epsilon)k}, 0, \dots, 0). \qquad (4.2.2)$$

The restriction of this representation to $O(a-1) \times O(1)$ is as follows. Restrict the representation of $U(a)$ with highest weight (4.2.2) to $U(a-1) \times U(1)$. The representations on $U(a-1)$ which correspond to irreducible representations of $O(a-1)$ as in (4.2.1-4.2.2) give the factors in the restriction. The character corresponding to an even integer on $U(1)$ gives a $(+)$, an odd one a $(-)$. We list the cases explicitly. Suppose $a = 2n$, and the highest weight is of the form

$$(a_1, \dots, a_n), \qquad a_n \ge 0. \qquad (4.2.3)$$

When $a_n > 0$, the representation of $O(2n)$ decomposes into two irreducible factors of the same dimension whne restricted to $SO(2n)$. Its restriction to $O(a-1) \times O(1)$ is $\sum_{a_{i+1} \le b_i \le a_i} (b_1, \dots, b_{n-1}; \epsilon) \otimes (1)$, with

$$\frac{1-\epsilon}{2} \equiv \sum a_i - \sum b_j (mod\ 2). \qquad (4.2.4)$$

In case $a_n = 0$, the restriction is $\displaystyle\sum_{a_{i+1}\le b_i\le a_i} (b_1,\dots,b_{n-1};\alpha)\otimes(\epsilon)$, with

$$\frac{1-\epsilon}{2}\equiv \sum a_i - \sum b_j + \frac{1-\alpha}{2}(mod\ 2). \tag{4.2.5}$$

When $a = 2n+1$, the restriction of $(a_1,\dots,a_n;\alpha)$ to $SO(2n+1)$ is irreducible. Its restriction to $O(2n)\times O(1)$ is

$$\sum_{a_{i+1}\le b_i\le a_i} (b_1,\dots,b_n;\alpha)\otimes(\epsilon)$$

with

$$\frac{1-\epsilon}{2}\equiv \sum a_i - \sum b_j + \frac{1-\alpha}{2}(mod\ 2). \tag{4.2.6}$$

In this formula we assume $a_{n+1} = 0$.

For $O(n,n)$ we use the K-types

$$(0,\dots,0;+)\otimes(\underbrace{2,\dots,2}_{r},0,\dots,0;+) \tag{4.2.7}$$

$$(\underbrace{1,\dots,1}_{k},0,\dots,0;\epsilon)\otimes(\underbrace{1,\dots,1}_{k},0,\dots,0;\epsilon) \qquad \epsilon=\pm. \tag{4.2.8}$$

The restriction of (4.2.8) to $S[O(n)\times O(n)]$ is independent of ϵ. We will show that the representations of $W(D_n)$ on V^M are irreducible. Precisely, the representation of W on V^M is

$$(r,n-r)\times(0) \qquad \longleftrightarrow (4.2.7) \tag{4.2.9}$$

$$(n-k)\otimes(k),\quad k\le[n/2] \qquad \longleftrightarrow (4.2.8)\text{ with }+, \tag{4.2.10}$$

$$(n-k)\otimes(k),\quad k>[n/2] \qquad \longleftrightarrow (4.2.8)\text{ with }-. \tag{4.2.11}$$

For $O(n+1,n)$ we use

$$(0,\dots,0;+)\otimes(\underbrace{2,\dots,2}_{r},0,\dots,0;+) \tag{4.2.12}$$

$$(\underbrace{1,\dots,1}_{k},0,\dots,0;+)\otimes(\underbrace{1,\dots,1}_{k},0,\dots,0;+) \qquad \text{for } k\le[n/2] \tag{4.2.13}$$

$$(\underbrace{1,\dots,1}_{n+1-k},0,\dots,0;-)\otimes(\underbrace{1,\dots,1}_{n-k},0,\dots,0;-) \qquad \text{for } n\ge k>[n/2] \tag{4.2.14}$$

The corresponding representations of $W(B_n)$ on V^M are

$$(r,n-r)\times(0) \qquad \longleftrightarrow (4.2.12) \tag{4.2.15}$$

$$(n-k)\otimes(k),\quad k\le[n/2] \qquad \longleftrightarrow (4.2.13)\text{ with }+, \tag{4.2.16}$$

$$(n-k)\otimes(k),\quad k>[n/2] \qquad \longleftrightarrow (4.2.14)\text{ with }-. \tag{4.2.17}$$

The group M is the diagonal group $\underbrace{O(1) \times \cdots \times O(1)}_{n+a}$ inside

$$\underbrace{O(1) \times \cdots \times O(1)}_{n+a} \times \underbrace{O(1) \dots O(1)}_{n}, \qquad a = 0, 1$$

where the first $O(1)$ is diagonal inside the 1'st and $n+a+1$'st and so on up to n and $2n$, and the $n+a$'th by itself.

Consider cases (4.2.8) and (4.2.16-4.2.17). The representations can be realized as $\Lambda^k \mathbb{C}^{n+a} \otimes \Lambda^k \mathbb{C}^n$. Let e_i be a basis of $\mathbb{C}^{n+a}$ and f_j a basis of $\mathbb{C}^n$. The space V^M is the span of the vectors $e_{i_1} \wedge \cdots \wedge e_{i_k} \otimes f_{i_1} \wedge \cdots \wedge f_{i_k}$. The representation of $W(B_n) \cong S_n \rtimes (\mathbb{Z}_2)^n$ on this space is the induced from the trivial representation on $S_k \times S_{n-k}$ tensored with the sign representation on $\mathbb{Z}_2^k$ and the trivial representation on $\mathbb{Z}_2^{n-k}$. This is precisely what (4.2.10-4.2.11) and (4.2.13-4.2.14) state.

For cases (4.2.7) and (4.2.12) we have to use the restriction rules from $O(n)$ to $O(n-1) \times O(1)$.

Formulas (4.2.9-4.2.15) are easily proved for $n = 1$. For the rest we do an induction.

Write $\mu_o(r)$ for the K-type

$$(0;+) \otimes (\underbrace{2, \dots, 2}_{r}, \underbrace{0, \dots, 0}_{n-r}; +) \tag{4.2.18}$$

In cases $O(2n+1, 2n)$ and $O(2n, 2n)$ the Weyl group is $W(B_{2n})$, while in the cases $O(2n, 2n-1)$ and $O(2n-1, 2n-1)$ it is $W(B_{2n-1})$. We only give details for part of the proof of the induction step. For $O(2n+1, 2n+1)$, the restriction of $\mu_o(r)$ to $O(2n) \times O(1) \times O(2n)$ is irreducible. The result follows from the induction hypothesis and the fact that M and $W(B_{2n})$ are contained in this subgroup. For $O(2n, 2n)$, the representation with $r = 0$ is trivial and the result holds. Thus assume $r > 0$. The Weyl group is $W(B_{2n})$. Then $\mu_o(r)$ restricts to $O(2n) \times O(2n-1) \times O(1)$ to

$$\mu_o(r) \otimes (+) \tag{4.2.19}$$

$$(\underbrace{2, \dots, 2}_{r-1}, 1, \underbrace{0, \dots, 0}_{n-r-1}) \otimes (0;+) \otimes (-) \tag{4.2.20}$$

$$\mu_o(r-1) \otimes (+) \tag{4.2.21}$$

Then M is contained in this subgroup and $O(2n) \times O(2n-1)$ is the maximal compact subgroup of $O(2n, 2n-1)$. So the induction hypothesis applies. Only (4.2.19) and (4.2.21) contribute to V^M and (4.2.19) does not occur when $r = n$. The intersection of $W(B_{2n})$ with this subgroup is $W(B_{2n-1})$. By the induction hypothesis, the representations of $W(B_{2n-1})$ on the M-fixed vectors of (4.2.19), (4.2.21) are

$$(r, 2n-1-r) \times (0), \;\; (r-1, 2n-r) \times (0). \tag{4.2.22}$$

If $r > 1$, then $(r-1, 2n-r)$ can only come from

$$(r, 2n) \times (0), \tag{4.2.23}$$
$$(1, r-1, 2n-r) \times (0), \tag{4.2.24}$$
$$(r-1, 2n+1) \times (0). \tag{4.2.25}$$

But (4.2.24) contains $(1, r-1, 2n-r-1) \times (0)$ in its restriction and (4.2.25) contains $(r-2, 2n+1-r) \times (0)$ in its restriction, so V^M must be (4.2.23) as claimed. If $r = 1$, $(0, 2n-1) \times (0)$ can only come from

$$(1, 2n-1) \times (0), \tag{4.2.26}$$
$$(0, 2n) \times (0). \tag{4.2.27}$$

If (4.2.27) occurs in V^M, then (4.2.26) has to occur as well to account for $(1, 2n-2) \times (0)$. But then $(0, 2n-2) \times (0)$ would occur more than once, a contradiction.

Remark. In the cases of type D, the Weyl group that is relevant for the calculations with the intertwining operators are type D. In formulas (4.2.10-4.2.11) for $k < [n/2]$, the Weyl group representations of $W(B_n)$ restrict to the same representation of $W(D_n)$, and there is no need for $k > [n/2]$. For $k = [n/2]$, the representation in (4.2.10), (4.2.11) decomposes into the sum of $(n) \times (n)_{I,II}$. But so does the restriction of the representation in (4.2.8) and each factor contains only one of the Weyl group representations.

4.3. **General linear groups.** The maximal compact subgroup of $GL(n, \mathbb{R})$ is $O(n)$, the Weyl group is $W(A_{n-1}) = S_n$ and group $M \cong \underbrace{O(1) \times \cdots \times O(1)}_{n}$.

In this case we use the K-types with highest weights

$$(\underbrace{2, \dots, 2}_{k}, 0, \dots, 0; +).$$

The corresponding Weyl group representations on V^M are $\mu(k, n-k) := (k, n-k)$. We omit the details.

4.4. **Unitary groups.** Let $U(p,q)$ be a unitary group with $p \geq q$. The maximal compact subgroups is $U(p) \times U(q)$, and we can identify

$$M \cong U(p-q) \times \underbrace{U(1) \times \cdots \times U(1)}_{q} \tag{4.4.1}$$

where eaqch $U(1)$ is embedded diagonally on the $p-q+i$ and $p+i$ entry. The Weyl group is $W(B_q)$. The relevant K-types are

$$\mu_e^+ = (\underbrace{1,\dots,1}_{k},0,\dots,0)\otimes(0,\dots,0,\underbrace{-1,\dots,-1}_{k}) \qquad k\le q, \tag{4.4.2}$$

$$\mu_e^- = (0,\dots,0,\underbrace{-1,\dots,-1}_{k})\otimes(\underbrace{1,\dots,1}_{k},0,\dots,0) \qquad k\le q, \tag{4.4.3}$$

$$\mu_o^- = (0,\dots,0)\otimes((\underbrace{1,\dots,1}_{k},0,\dots,0,\underbrace{-1,\dots,-1}_{k}) \qquad k\le [\frac{q}{2}]. \tag{4.4.4}$$

We will suppress the $\pm$ superscripts; the μ_e behave the same way and there is only one μ_o that we will consider. The same argument as for $Sp(n)$ shows that V^M is an irreducible representation representation of W. The correspondence is

$$\mu_e(k,q-k)\longleftrightarrow(q-k)\times(k), \tag{4.4.5}$$

$$\mu_o(k,q-k)\longleftrightarrow(k,q-k)\times(0). \tag{4.4.6}$$

4.5. **Orthogonal groups.** Let $O(p,q)$ be an orthogonal group with $p>q+1$. The maximal subgroup is $O(p)\times O(q)$ and we can identify

$$M\cong O(p-q)\times\underbrace{O(1)\times\cdots\times O(1)}_{q} \tag{4.5.1}$$

with the $O(1)$'s embedded diagonally just as in the case of the unitary groups. The Weyl group is of type $W(B_q)$. We use the same relevant K-types as in the case of the split orthogonal groups. The same arguments show that

$$\mu_e(k,q-k)\leftrightarrow(q-k)\times(k), \qquad k\le q, \tag{4.5.2}$$

$$\mu_o(k,q-k)\leftrightarrow(k,q-k)\times(0), \qquad k\le[\frac{q}{2}]. \tag{4.5.3}$$

4.6. **Summary.** In type A and types $B,\ C,\ D$ for $(m, n-m)\times(0)$ the condition $m \le [\frac{n}{2}]$ has to hold.

Type	V	V^M
AI	$(\underbrace{2,\dots,2}_{m},0,\dots,0;+)$	$\mu(m,n-m)$,
B, D	$(\underbrace{1,\dots,1}_{m},0,\dots,0;+)\otimes(\underbrace{1,\dots,1}_{m},0,\dots,0;+)$	$(n-m)\times(m)$,
	$(\underbrace{0,\dots,0};0+)\otimes(\underbrace{2,\dots,2}_{m},\underbrace{0,\dots,0};0+)$	$(m,n-m)\times(0)$,
$AII,\ C$	$(\underbrace{2,\dots,2}_{m},0,\dots,0)$	$(n-m)\times(m)$,
	$(\underbrace{1,\dots,1}_{m},0,\dots,0,\underbrace{-1,\dots,-1}_{m})$	$(m,n-m)\times(0)$.

Definition. *The above K-types will be called relevant. Denote them by*

$$\begin{aligned}\mu(m,n-m)&:=(m,n-m),\\ \mu_e(m,n-m)&:=(n-m)\times(m),\\ \mu_o(m,n-m)&:=(m,n-m)\times(0).\end{aligned}$$

We will write $\mu_e(m)$ and $\mu_o(m)$ when there is no danger of confusion what n is.

5. Intertwining Operators

5.1. Let $w \in W$. Then there is an intertwining operator

$$I(w,\nu): X(\nu)\longrightarrow X(w\nu). \tag{5.1.1}$$

If (μ, V) is a K-type, then I induces a map

$$I_V(w,\nu):\mathrm{Hom}_K[V,X(\nu)]\longrightarrow \mathrm{Hom}_K[V,X(w\nu)]. \tag{5.1.2}$$

By Frobenius reciprocity, we get a map

$$R_V(w,\nu):(V^*)^M\longrightarrow (V^*)^M. \tag{5.1.3}$$

In case (μ, V) is trivial the spaces are 1-dimensional and $I_V(w,\nu)$ is a scalar. We normalize $I(w,\nu)$ so that this scalar is 1. The $R_V(w,\nu)$ are meromorphic functions in ν, and the $I(w,\nu)$ have the following additional properties.

(1) If $w = w_1\cdot w_2$ with $\ell(w)=\ell(w_1)+\ell(w_2)$, then $I(w,\nu)=I(w_1,w_2\nu)\circ I(w_2,\nu)$. In particular if $w = s_{\alpha_1}\cdots s_{\alpha_k}$ is a reduced decomposition,

then $I(w)$ factors into a product of intertwining operators I_j, one for each s_{α_j}. These operators are

$$I_j \;:\; X(s_{\alpha_{j+1}} \dots s_{\alpha_k} \cdot \nu) \longrightarrow X(s_{\alpha_j} \dots s_{\alpha_k} \cdot \nu) \tag{5.1.4}$$

(2) Let $P = MN$ be a standard parabolic subgroup (so $A \subset M$) and $w \in W(M, A)$. The intertwining operator

$$I(w,\nu) \;:\; X(\nu) = Ind_P^G[X_M(\nu)] \longrightarrow X(w\nu) = Ind_P^G[X_M(w\nu)]$$

is of the form $I(w,\nu) = Ind_M^G[I_M(w,\nu)]$.

(3) If $Re\langle\nu,\alpha\rangle \geq 0$ for all positive roots α, then $R_V(w_0,\nu)$ has no poles, and the image of $I(w_0,\nu)$ ($w_0 \in W$ is the long element) is $L(\nu)$.

(4) If $-\overline{\nu}$ is in the same Weyl group orbit as ν, let w be the shortest element so that $w\nu = -\overline{\nu}$. Then $L(\nu)$ is hermitian with inner product

$$\langle v_1, v_2\rangle := \langle v_1, I(w,\nu)v_2\rangle.$$

Let α be a simple root and $P_\alpha = M_\alpha N$ be the standard parabolic subgroup so that the Lie algebra of M_α is isomorphic to the $sl(2,\mathbb{R})$ generated by the root vectors $E_{\pm\alpha}$. We assume that $\theta E_\alpha = -E_{-\alpha}$. Let $D_\alpha = \sqrt{-1}(E_\alpha - E_{-\alpha})$ and $s_\alpha = e^{\sqrt{-1}\pi D_\alpha/2}$. Then $s_\alpha^2 = m_\alpha$ is in $M \cap M_\alpha$. Since the square of any element in M is in the center and M normalizes the the root vectors, $\mathrm{Ad}\, m(D_\alpha) = \pm D_\alpha$. Grade $V^* = \oplus V_i^*$ according to the absolute values of the eigenvalues of D_α (which are integers). Then M preserves this grading and

$$(V^*)^M = \bigoplus_{i\ even} (V_i^*)^M.$$

The map $\psi_\alpha : sl(2,\mathbb{R}) \longrightarrow \mathfrak{g}$ determined by

$$\psi_\alpha \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = E_\alpha, \qquad \psi_\alpha \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = E_{-\alpha}$$

determines a map

$$\Psi_\alpha \;:\; SL(2,\mathbb{R}) \longrightarrow G \tag{5.1.5}$$

with image G_α, a connected group with Lie algebra isomorphic to $sl(2,\mathbb{R})$. Let R_α be the maps (5.1.3) for G_α.

Proposition. *On* $(V_{2m}^*)^M$,

$$R_V(s_\alpha,\nu) = \begin{cases} Id & \text{if } m = 0, \\ \prod_{0\leq j<m} \frac{2j+1-<\nu,\check{\alpha}>}{2j+1+<\nu,\check{\alpha}>}\; Id & \text{if } m \neq 0. \end{cases}$$

In particular, $I(w,\nu)$ *is an isomorphism unless* $\langle\nu,\check{\alpha}\rangle \in -\mathbb{N}$.

Proof. The formula is well known for $SL(2,\mathbb{R})$. The second assertion follows from this and the listed properties of intertwining operators. □

Corollary. *For relevant K-types the formula is*

$$R_V(s_\alpha,\nu) = \begin{cases} Id & \text{on the +1 eigenspace of } s_\alpha, \\ \frac{1-<\nu,\check{\alpha}>}{1+<\nu,\check{\alpha}>}\; Id & \text{on the -1 eigenspace of } s_\alpha. \end{cases}$$

When restricted to $(V^*)^M$, *the long intertwining operator is the product of the* R_α *corresponding to the reduced decomposition of* w_0 *and depends only on the Weyl group structure of* $(V^*)^M$.

Proof. Relevant K-types are distinguished by the property that the eigenvalues of D_α are $0, \pm 2$ only. The element s_α acts by 1 on the zero eigenspace of D_α and by -1 on the ± 2 eigenspace. The claim follows from this. □

5.2. We now show that the formulas in the previous section coincide with corresponding ones in the p-adic case. Recall from [BM3] that the induced module is $X(\nu) := \mathbb{H} \otimes_{\mathbb{A}} \mathbb{1}_\nu$ where $\mathbb{H} = \mathbb{C}[W] \times \mathbb{A}$ (corresponding to the dual root system) is the graded affine Hecke algebra. The abelian subalgebra $\mathbb{A}$ is generated by $\omega \in S(\mathfrak{a})$ ($\mathfrak{a} = Lie(A)$) and $\mathbb{C}[W]$ is generated by $\{t_\alpha\}_{\alpha\ simple}$ satisfying $t_\alpha^2 = 1$. They are subject to the relations

$$\omega t_\alpha = s_\alpha(\omega) t_\alpha + c_\alpha < \omega, \check{\alpha} >, \qquad \omega \in S(\mathfrak{a}). \tag{5.2.1}$$

The scalars c_α are assumed invariant under the action of the Weyl group on the roots. The intertwining operator $I(w,\nu)$ is a product of operators I_{α_i} according to a reduced decomposition of $w = s_{\alpha_1} \cdot \dots \cdot s_{\alpha_k}$. If α is a simple root,

$$r_\alpha := (t_\alpha \alpha - c_\alpha)\frac{1}{\alpha - c_\alpha}, \qquad I_\alpha \ : \ x \otimes \mathbb{1}_\nu \mapsto x r_\alpha \otimes \mathbb{1}_{s_\alpha \nu}. \tag{5.2.2}$$

We normalize the c_α so that $c_\alpha = 1$ for the roots of the form $\epsilon_i \pm \epsilon_j$. We only need to consider type A and D with $c_\alpha = 1$ and type B with $c_\alpha = c$ arbitrary for α a short root. This is because type C is equivalent to type B by setting $c_\alpha = c/2$ for the long roots.

We consider the split cases first. Then $c_\alpha = 1$ for all roots. The $I(w,\nu)$ have the same properties as in the real case. The r_α are multiplied on the right, so we can replace α with $-\langle \nu, \alpha \rangle$ in the formulas. Furthermore,

$$\mathbb{C}[W] = \sum_{\sigma \widehat{W}} V_\sigma \otimes V_\sigma^*.$$

Since r_α acts as multiplication on the right, it gives rise to an operator

$$r_\sigma(s_\alpha, \nu) : V_\sigma^* \longrightarrow V_\sigma^*.$$

Theorem. *The* $R_V(s_\alpha, \nu)$ *for the real case on relevant* K*-types coincide with the* $r_\sigma(s_\alpha, \nu)$ *on the* $V_\sigma^* \cong (V^*)^M$

Proof. The operators R_α and r_α act the same way:

$$r_\sigma(s_\alpha, \nu) = \begin{cases} Id & \text{on the } +1 \text{ eigenspace of } t_\alpha \\ \frac{1-\langle \nu, \alpha \rangle}{1+\langle \nu, \alpha \rangle} Id & \text{on the } -1 \text{ eigenspace of } t_\alpha \end{cases} \tag{5.2.3}$$

The assertion is now clear from corollary (5.1) and formula (5.2.2). We emphasize that the Hecke algebra is for the dual root system so that there is no discrepancy between α and $\check{\alpha}$ in the formulas. □

5.3. We identify the relevant K-types with the corresponding Weyl group representations. Recall that a *special unipotent* parameter is given by an even nilpotent in the dual algebra as in sections 1.2. Let χ_0 be the infinitesimal character. We attach two parabolic subgroups to such a parameter, P_e and P_o with Levi components

$$\begin{array}{ll}
\mathbf{M_e}: & \\
B & GL(x_{2m-1}+x_{2m-2})\times\cdots\times GL(x_1+x_0)\times G(x_{2m}) \\
C & GL(x_{2m-1}+x_{2m-2}+1)\times\cdots\times GL(x_1+x_0+1)\times G(x_{2m}) \\
D & GL(x_{2m-1}+x_{2m-2}+1)\times\cdots\times GL(x_1+x_0+1) \\
 & \\
\mathbf{M_o}: & \\
B & GL(x_{2m}+x_{2m-1})\times\cdots\times GL(x_2+x_1)\times G(x_0) \\
C & GL(x_{2m}+x_{2m-1}+1)\times\cdots\times GL(x_2+x_1+1)\times G(x_0) \\
D & GL(x_{2m-3}+x_{2m-4}+1)\times\cdots\times GL(x_{2m-2})\times G(x_{2m-1})
\end{array} \tag{5.3.1}$$

For each there is a 1-dimensional representation χ_e and χ_o such that the spherical irreducible representation $L(\chi_{\check{\mathcal{O}}})=\overline{X}(\chi_{\check{\mathcal{O}}})$ with infinitesimal character χ_0 is the spherical irreducible subquotient of $X_e := Ind_{P_e}^G(\chi_e)$ and $X_o := Ind_{P_o}^G(\chi_o)$ respectively. Precisely each character corresponds to a string which we write in decreasing order. This corresponds to the action of $S(\mathfrak{a})$ and also means that X_e is a quotient of the standard module with the same parameter. Precisely the strings are

$$\dots(-x_{2i-1}+\epsilon,\dots,x_{2i-2}-\epsilon)\dots(-x_{2m}+\epsilon,\dots,1-\epsilon) \tag{5.3.2}$$

for X_e and

$$\dots(-x_{2i}+\epsilon,\dots,x_{2i-1}-\epsilon)\dots(-x_0+\epsilon,\dots,1-\epsilon)\ \text{type B,C} \tag{5.3.3}$$

$$\begin{gathered}\dots(-x_{2i}+\epsilon,\dots,x_{2i-1}-\epsilon)\dots(-x_{2m-2}+1,\dots,0)\\ (-x_{2m-1}+1,\dots,0)\ \text{type D}\end{gathered} \tag{5.3.4}$$

for X_o.

Theorem. *The relations*

$$[\mu_e(m)\ :\ X_e]=[\mu_e(m)\ :\ L(\chi_0)],\qquad [\mu_o(m)\ :\ X_o]=[\mu_o(m)\ :\ L(\chi_0)]$$

hold for special unipotent parameters.

The proof is in section 6.7.

For a general parameter, the strings defined in section 2 and the above construction defines parabolic subgroups with Levi components $M_e\times GL(k_1)\times\cdots\times GL(k_r)$ and $M_o\times GL(k_1)\times\cdots\times GL(k_r)$ and characters on each factor. The $GL(k_i)$ come from the remaining strings in 2.3 We denote the induced modules by X_e and X_o as well.

Corollary. *The relations*

$$[\mu_e(m)\ :\ X_e]=[\mu_e(m)\ :\ L(\chi)],\qquad [\mu_o(m)\ :\ X_o]=[\mu_o(m)\ :\ L(\chi)]$$

hold in general.

The proof is in section 6.8.

6. Hecke algebra calculations

6.1. The idea of the proof of the results in 5.3 is very simple, but notation is rather cumbersome. We try to simplify it as much as possible. We work in the setting of the Hecke algebra, but the results hold for the real groups as well. We will write $GL(k)$ for the Hecke algebra of type A and $G(n)$ for the one of type B, C or D as the case may be. We use the term K-type for a representation of the corresponding Weyl group, and induced modules as in (6.1.5) are of the form $X_P(\nu)=\mathbb{H}\otimes_{\mathbb{H}_M}[\mathbb{C}_\nu\otimes triv]$.

Suppose $P=MN$ is a standard parabolic subgroup with Levi component

$$GL(k_1)\times\dots\times GL(k_l)\times G(n). \tag{6.1.1}$$

Let χ_i be characters for $GL(k_i)$. We write

$$\chi_i\longleftrightarrow(\nu_i):=(-\frac{k_i-1}{2}+\nu_i,\dots,\frac{k_i-1}{2}+\nu_i). \tag{6.1.2}$$

This has the property that the representation χ_i occurs as a submodule of $X(\nu_i)$. The action of the Hecke algebra however is

$$\chi_i(\omega)=\langle\omega,(\frac{k_i-1}{2}+\nu_i,\dots,-\frac{k_i-1}{2}+\nu_i)\rangle,\qquad \omega\in\mathfrak{a}. \tag{6.1.3}$$

In this notation the trivial representation of $G(n)$ corresponds to the string

$$(\nu_n):=\begin{cases}(-n+1-c,\dots,-c+\epsilon), & \text{type B},\\ (-n+1,\dots,0), & \text{type D}.\end{cases} \tag{6.1.4}$$

We write

$$X_P(\dots(\nu_i)\dots):=Ind^G_{\prod GL(k_i)\times G(n)}[\otimes\chi_i\otimes triv]. \tag{6.1.5}$$

The subscript indicates that the module is induced from a parabolic subgroup. The Levi components can be read off from the string. The module $X_P(\dots(\nu_i)\dots)$ is a submodule of the standard module with parameter corresponding to the strings

$$\nu:=(\dots,-\frac{k_i-1}{2}+\nu_i,\dots,\frac{k_i-1}{2}+\nu_i,\dots,-n-c+\epsilon,\dots,-1-c+\epsilon). \tag{6.1.6}$$

Let $w_i\in W$ be the shortest Weyl group element which interchanges the strings (ν_i) and (ν_{i+1}) in ν, and fixes all other entries. The intertwining operator $I_{w_i}:X(\nu)\longrightarrow X(w_i\nu)$ restricts to an intertwining operator

$$I(\nu_i,\nu_{i+1}):X_P(\dots(\nu_i)(\nu_{i+1})\dots)\longrightarrow X_P(\dots(\nu_{i+1})(\nu_i)\dots). \tag{6.1.7}$$

This operator is induced from the similar one on $GL(k_i+k_{i+1})$ where $M=GL(k_i)\times GL(k_{i+1})$ is the Levi component of a maximal parabolic subgroup.

Let $w_l \in W$ be the shortest element which changes ν_l to $-\nu_l$, and fixes all other entries. It induces an intertwining operator

$$I(\nu_l) : X_P(\dots(\nu_l),(\nu_n)) \longrightarrow X_P(\dots(-\nu_l),(\nu_n)). \qquad (6.1.8)$$

In type D the last entry of the resulting string might have to stay $-\frac{k_l-1}{2}+\nu_l$. This operator is induced from the similar one on $G(k_l+n)$ where $M = GL(k_l) \times G(n)$ is the Levi component of a maximal parabolic subgroup.

Let μ be a K-type. We are interested in computing the matrices $r_\mu(w_0,\nu)$ from section 5. They can be factored into terms of the type (6.1.7) and (6.1.8). To compute these, the main tool is Frobenius reciprocity. Let $P' = M'N'$ be the standard parabolic subgroup with Levi component M'

$$GL(k_1) \times \dots \times GL(k_i + k_{i+1}) \times \dots \text{ in case (6.1.7)} \qquad (6.1.9)$$

$$GL(k_1) \times \dots \times G(k_l + n) \text{ in case (6.1.8)} \qquad (6.1.10)$$

In the real case, the relevant K-types are identified with the corresponding Weyl group representations and in the p-adic case they are Weyl group representations to begin with. We have

$$\mathrm{Hom}_W[\mu, X((\nu_i))] = Hom_{W(M')}[\mu|_{W(M')} \ : \ triv \otimes X((\nu_i),(\nu_{i+1})) \otimes triv] \quad \text{in case (6.1.7)} \qquad (6.1.11)$$

$$\mathrm{Hom}_W[\mu, X((\nu_i))] = Hom_{W(M')}[\mu|_{W(M')} \ : \ triv \otimes X((\nu_l),(\nu_n))] \quad \text{in case (6.1.8)} \qquad (6.1.12)$$

The restrictions of relevant K-types to Levi components consists of relevant K-types of the same kind. We write

$$\mu \longrightarrow \sum m_i\mu_i \qquad (6.1.13)$$

for the K-types μ_i with multiplicities m_i in $X((\nu_i),(\nu_{i+1}))$ or $X((\nu_l),(\nu_n))$ that figure in formulas (6.1.11) and (6.1.12). In general the multiplicities are 1. The matrix r_μ is then computed from the corresponding scalars for the μ_i. Theorem 5.3 and corollary 5.3 depend heavily on computing the matrices r_μ for μ relevant and the intertwining operators I_i.

6.2. $GL(a) \times GL(b)$**.** This is the case of I_i with $i < l$. Let $n = a + b$ and $G = GL(n)$ and $P = MN$ be the standard parabolic subgroup with Levi component $GL(a) \times GL(b)$. The module $X_P((\nu_1),(\nu_2))$ induced from the characters corresponding to

$$(-\frac{a-1}{2}+\nu_1,\dots,\frac{a-1}{2}+\nu_1),(-\frac{b-1}{2}+\nu_2,\dots,\frac{b-1}{2}+\nu_2) \qquad (6.2.1)$$

has the following S_{a+b} structure. Let $m := \min(a,b)$ and write $\mu(k,a+b-k)$ for the module corresponding to the partition $(k, a+b-k)$, $0 \le k \le m$. Then

$$X_P((\nu_1),(\nu_2)) = \bigoplus_{0\le k\le m} \mu(k,a+b-k). \qquad (6.2.2)$$

Lemma. *For $1 \leq k \leq m$, the intertwining operator $I((\nu_1)(\nu_2))$ has*

$$r_{\mu(k,a+b-k)}(a,b,\nu_1,\nu_2) = \prod_{0\leq j\leq k-1} \frac{(\nu_1 - \frac{a-1}{2}) - (\frac{b-1}{2} + \nu_2 + 1) + j}{(\nu_1 + \frac{a-1}{2}) - (-\frac{b-1}{2} + \nu_2 - 1) - j}.$$

Proof. The proof is an induction on a, b and k. We omit most details but give the general idea. Assume $k < m$, the case $k = m$ is simpler. Embed $X_P((\nu_1),(\nu_2))$ into $X_P((\nu'),(\nu''),(\nu_2))$ corresponding to the strings

$$(-\frac{a-1}{2} + \nu_1, \dots, \frac{a-3}{2} + \nu_1)(\frac{a-1}{2} + \nu_1), (-\frac{b-1}{2} + \nu_2, \dots, \frac{b-1}{2} + \nu_2). \tag{6.2.3}$$

The intertwining operator $I(\nu_1,\nu_2)$ is the restriction of

$$I_1(\nu',\nu_2,\nu'') \circ I_2(\nu',\nu'',\nu_2) \tag{6.2.4}$$

to $X_P((\nu_1),(\nu_2))$, where I_2 interchanges the strings $(\nu''),(\nu_2)$ and I_1 interchanges $(\nu'),(\nu_2)$ and they each fix the remaining one. The K-type $\mu(k,n-k)$ occurs with multiplicity 1 in $X_P(\nu_1),(\nu_2))$ and with multiplicity 2 in $X((\nu'),(\nu''),(\nu_2))$. The restrictions are

$$\mu(k,n-k) \longrightarrow triv \otimes \mu(k-1,b+1-k) + triv \otimes \mu(k,b-k) \text{ for } I_1 \tag{6.2.5}$$

$$\mu(k,n-k) \longrightarrow \mu(1,b) + \mu(0,b+1) \text{ for } I_2 \tag{6.2.6}$$

The representation $\mu(k,n-k)$ has a realization as harmonic polynomials in $S(\mathfrak{a})$ spanned by

$$\prod_{1\leq l\leq k} (\epsilon_{i_l} - \epsilon_{j_l}) \tag{6.2.7}$$

where $(i_1,j_1),\dots,(i_\ell,j_\ell)$ are ℓ pairs of integers $i_k \neq j_k$, and $1 \leq i_k,\ j_k \leq n$. We apply the intetwining operator to the $S_a \times S_b$-fixed vector

$$e := \sum_{\sigma\in S_a\times S_b} \sigma \cdot [(\epsilon_1 - \epsilon_{a+1}) \times \dots \times (\epsilon_k - \epsilon_{a+k})]. \tag{6.2.8}$$

The intertwining operator I_2, has a simple form on the vectors

$$\mu(0,b+1) \leftrightarrow \tag{6.2.9}$$

$$e_1 := \sum_{\sigma\in S_{a-1}\times S_{b+1}} \sigma \cdot [(\epsilon_1 - \epsilon_{a+1}) \times \dots \times (\epsilon_k - \epsilon_{a+k})],$$

$$\mu(1,b) \leftrightarrow \tag{6.2.10}$$

$$e_2 := \sum_{\sigma\in S_{a-1}\times S_1\times S_b} \sigma \cdot [(\epsilon_1 - \epsilon_{a+1}) \times \dots \times (\epsilon_{k-1} - \epsilon_{a+k-1})(\epsilon_a - \epsilon_{a+k})],$$

transforming according to (6.2.6). They are mapped into scalar multiples (given by the lemma) of the vectors $e_1',\ e_2'$ which are invariant under $S_{a-1} \times$

$S_b \times S_1$, and transform according to $triv \otimes \mu(0, b+1)$ and $triv \otimes \mu(1, b)$. We choose

$$\begin{aligned} e_1' &= e_1, \\ e_2' &:= \sum_{\sigma \in S_{a-1} \times S_b \times S_1} \sigma \cdot [(\epsilon_1 - \epsilon_a) \times \cdots \times (\epsilon_{k-1} - \epsilon_{a+k-2})(\epsilon_n - \epsilon_{a+k-1})] \end{aligned} \tag{6.2.11}$$

The intertwining operator I_1 has a simple form on the vectors invariant under $S_{a-1} \times S_b \times S_1$ transforming according to $\mu(k, n-k-1)$ and $\mu(k-1, n-k)$. We can choose multiples of

$$\mu(k-1, n-k) \leftrightarrow f_1 := \tag{6.2.12}$$
$$\sum_{\sigma \in S_{a-1} \times S_b \times S_1} \sigma[(\epsilon_1 - \epsilon_a) \times \cdots \times (\epsilon_{k-1} - \epsilon_{a+k-2})(\epsilon_k - \epsilon_{a+k-1})],$$
$$\mu(k, n-k-1) \leftrightarrow f_2 := \tag{6.2.13}$$
$$\sum_{\sigma \in S_{a-1} \times S_b \times S_1} \sigma[(\epsilon_1 - \epsilon_a) \times \cdots \times (\epsilon_{k-1} - \epsilon_{a+k-2}) \cdot$$
$$\cdot (e_k + \cdots + \epsilon_{a-1} + \epsilon_a + \epsilon_{a+k} + \cdots + \epsilon_{n-1} - (n-2k+1)\epsilon_n)].$$

The fact that f_1 transforms according to $\mu(k, n-1)$ follows from (6.2.7). The fact that f_2 transforms according to $\mu(k-1, n)$ is slightly more complicated. The product $\prod(\epsilon_1 - \epsilon_a) \times \cdots \times (\epsilon_{k-1} - \epsilon_{a+k-2})$ transforms according to $\mu(k-1, k-1)$ under S_{2k-2}. The vector $(e_k + \cdots + \epsilon_{a-1} + \epsilon_a + \epsilon_{a+k} + \cdots + \epsilon_{n-1} - (n-2k+1)\epsilon_n)$ is invariant under the S_{n-2k-1} acting on the coordinates $\epsilon_k, \dots \epsilon_a, \epsilon_{a+k}, \dots, \epsilon_{n-1}$. Since $\mu(k, n-k-1)$ does not have any vectors transforming this way, the product inside the sum in (6.2.13) must transform according to $\mu(k-1, n-k)$. The average under σ is nonzero. The operator I_2 maps f_1 and f_2 into multiples (using the induction hypothesis) of the vectors f_1', f_2' which are the $S_b \times S_{a-1} \times S_1$ invariant vectors transforming according to $\mu(k, n-1)$ and $\mu(k-1, n-k)$. The composition $I_1 \circ I_2$ maps e into a multiple of

$$e' := \sum_{\sigma \in S_b \times S_a} \sigma \cdot [(\epsilon_1 - \epsilon_{b+1}) \times \cdots \times (\epsilon_k - \epsilon_{b+k})]. \tag{6.2.14}$$

The multiple is computable by using the induction hypothesis and the expression of

e in terms of e_1, e_2,
e_1', e_2' in terms of f_1, f_2, and
e' in terms of f_1', f_2'.

For the case $k=1$, we get the following formulas.

$$\begin{aligned}
e &= b(\epsilon_1+\dots+\epsilon_a)-a(\epsilon_{a+1}+\dots+\epsilon_n),\\
e_1 &= (b+1)(\epsilon_1+\dots+\epsilon_{a-1})-(a-1)(\epsilon_a+\dots+\epsilon_n),\\
e_2 &= b\epsilon_a-(\epsilon_{a+1}+\dots+\epsilon_n),\\
f_1 &= b(\epsilon_1+\dots+\epsilon_{a-1})-(a-1)(\epsilon_a+\dots+\epsilon_{n-1}),\\
f_2 &= (\epsilon_1+\dots+\epsilon_{a-1})+(\epsilon_a+\dots+\epsilon_{n-1})-(n-1)\epsilon_n,\\
e' &= -a(\epsilon_1+\dots+\epsilon_b)-b(\epsilon_{b+1}+\dots+\epsilon_n),\\
e_1' &= (b+1)(\epsilon_1+\dots+\epsilon_{a-1})-(a-1)(\epsilon_a+\dots+\epsilon_n),\\
e_2' &= -(\epsilon_a+\dots+\epsilon_{n-1})+b(\epsilon_n),\\
f_1' &= -(a+1)(\epsilon_1+\dots+\epsilon_b)+b(\epsilon_{b+1}+\dots+\epsilon_{n-1}),\\
f_2' &= (\epsilon_1+\dots+\epsilon_b)+(\epsilon_{b+1}+\dots+\epsilon_{n-1})-(n-1)\epsilon_n.
\end{aligned} \tag{6.2.15}$$

The required formulas are

$$\begin{aligned}
e &= \frac{a-1}{b+1}e_1-\frac{n}{b+1}e_2,\\
e_1' &= \frac{n}{n-1}f_1+\frac{a-1}{n-1}f_2,\\
e_2' &= \frac{1}{n-1}f_1-\frac{b}{n-1}f_2,\\
e' &= \frac{n}{n-1}f_1'-\frac{b}{n-1}f_2'.
\end{aligned} \tag{6.2.16}$$

□

6.3. $GL(k)\times G(n)$. In the next two sections we prove theorem 5.3 in the case of a parabolic subgroup with Levi component $GL(k)\times G(n)$ for the induced module

$$X_P((\nu)) = Ind_P^G[\chi_\nu\otimes triv]. \tag{6.3.1}$$

The parameter corresponding to the character $\chi_\nu\otimes triv$ is

$$(-\frac{k-1}{2}+\nu,\dots,\frac{k-1}{2}+\nu)(-n+1-c+\epsilon,\dots,-c+\epsilon). \tag{6.3.2}$$

Recall that $\epsilon=0$ when the group is type B, $\epsilon=1$ for type D. Because of sections 5.1-5.3 we deal with r_σ. Then

$$I(\nu): X_P(\nu)\longrightarrow X_P(-\nu),\qquad X_P(\nu)=\sum V_\sigma\otimes (V_\sigma^*)^{W(M)}, \tag{6.3.3}$$

and $I(\nu)$ gives rise to an operator

$$r_\sigma(\nu): (V_\sigma^*)^{W(M)}\longrightarrow (V_\sigma^*)^{W(M)}. \tag{6.3.4}$$

We will compute $r_\sigma(\nu)$ by induction on k. In this case the relevant K-types have multiplicity ≤ 1 so $I(\nu)$ is a scalar. By embedding X_P into a bigger induced module we will decompose $r_\sigma(\nu)$ according to a reduced decomposition, compute the individual terms and then restrict their composition to the $W(M)$ fixed vectors of V^*.

6.4. We start with the special case $k = 1$ when the maximal parabolic subgroup P has Levi component $M = GL(1) \times G(n) \subset G(n+1)$. In type D we assume $n \geq 1$. The relevant K-types with multiplicities are

$$[(n+1) \times (0)] \ + \ [(1,n) \times (0)] \ + \ [(n) \times (1)]. \tag{6.4.1}$$

The operator $r_\sigma(\nu)$ can be written as a product

$$r_{1,2} \circ \cdots \circ r_{n,n+1} \circ r_{n+1} \circ r_{n,n+1} \circ \cdots \circ r_{1,2} \tag{6.4.2}$$

where $r_{i,j}$ is the r_σ corresponding to the root $\epsilon_i - \epsilon_j$ and r_{n+1} is the r_σ corresponding to ϵ_{n+1}. $I(\nu)$ is a rational function $f(\sigma,\nu)$ on each K-type σ, satisfying the relation $f(\sigma,-\nu)f(\sigma,\nu) = 1$.

Proposition. *The function $f(\sigma,\nu)$ equals*

$$\begin{array}{lll} & \mu_{\mathbf{e}}(\mathbf{1},\mathbf{n}) = (\mathbf{n}) \times (\mathbf{1}) & \mu_{\mathbf{o}}(\mathbf{1},\mathbf{n}) = (\mathbf{1},\mathbf{n}) \times (\mathbf{0}) \\ B & \frac{c+n-\nu}{c+n+\nu} & \frac{c+n-\nu}{c+n+\nu} \cdot \frac{c+\nu}{c-\nu} \\ C & \frac{n+1-\nu}{n+1+\nu} & -\frac{n+1-\nu}{n+1+\nu} \\ D & \frac{n-\nu}{n+\nu} & \frac{n-\nu}{n+\nu}\frac{1-\nu}{1+\nu} \end{array} \tag{6.4.3}$$

In this table the type refers to the group (not the Hecke algebra) and type C refers to the split case, i.e. $c = 1$.

Proof. We do an induction on n.

The reflection representation $(n) \times (1)$ has dimension $n+1$ and the usual basis $\{\epsilon_i\}$. The $W(M)$ fixed vector is ϵ_1. The representation $(1,n) \times (0)$ has a basis $\epsilon_i^2 - \epsilon_j^2$ with the symmetric square action. The $W(M)$ fixed vector is $\epsilon_1^2 - \frac{1}{n}(\epsilon_2^2 + \cdots + \epsilon_{n+1}^2)$.

The case $n = 0$ for type B is clear; the intertwining operator is 1 on $\mu_o = triv$ and $\frac{c-\nu}{c+\nu}$ on $\mu_e = sgn$. In type D and $n = 1$, the middle K-type in (6.4.1) decomposes further

$$[(2) \times (0)] + [(1) \times (1)_I] + [(1) \times (1)_{II}] + [(0) \times (2)]. \tag{6.4.4}$$

The representations $[(1) \times (1)_{I,II}]$ are 1-dimensional with bases $\epsilon_1 \pm \epsilon_2$. The result is clear in this case as well.

We now do the induction step. In the case μ_e, embed X_P in the induced module from the characters corresponding to

$$(\nu)(-n+1-c)(-n+2-c,\dots,-c). \tag{6.4.5}$$

Write $P' = M'N'$ for the standard parabolic subgroup corresponding to these three strings. Then the intertwining operator $I : X_P((\nu)) \longrightarrow X_P((-\nu))$ is the restriction of

$$I_1(-n+1-c,-\nu) \circ I_2(\nu) \circ I_1(\nu,-n+1-c). \tag{6.4.6}$$

In terms of the r_σ we get

$$(r_\sigma)_1(\nu,-n+1-c) \circ (r_\sigma)_2(\nu) \circ (r_\sigma)_1(-n+1-c,-\nu). \tag{6.4.7}$$

We need to compute the r_σ. For this we need to compute some restrictions of $\mu_e(1,n)$ and on $\mu_o(1,n)$. We have

$$\begin{aligned}
&Ind_{W(B_{n-1})}^{W(B_{n+1})}[(n-1)\times(0)] = (n+1)\times(0) + 2\ (n)\times(1) + (1,n)\times(0)\\
&\qquad + (1,n-1)\times(1) + (n-1)\times(2) + (n-1)\times(1,1), && \text{(a)}\\
&Ind_{W(B_n)}^{W(B_{n+1})}[(n)\times(0)] = (n+1)\times(0) + (n)\times(1) && \text{(b)}\\
& && (6.4.8)\\
&Ind_{W(A_1)W(B_n)}^{W(B_{n+1})}[(2)\otimes(n-1)\times(0)] = (n+1)\times(0) + (1,n)\times(0) +\\
&\qquad + (2,n)\times(0) + (n)\times(1) + (1,n-1)\times(1) + (n-1)\times(2) && \text{(c)}\\
&Ind_{W(A_1)W(B_n)}^{W(B_{n+1})}[(1,1)\otimes(n-1)\times(0)] = (1,n)\times(0) + (1,1,n)\times(0) +\\
&\qquad (n)\times(1) + (1,n)\times(1) + (n-1)\times(1,1) && \text{(d)}
\end{aligned}$$

Thus $\mu_e(1,n)$ occurs with multiplicity 2 in $X_{P'}$. The $W(M')$ fixed vectors are the linear span of $\epsilon_1,\ \epsilon_2$. The intertwining operators I_1 and I_2 are induced from maximal parabolic subgroups whose Levi components we label M_1 and M_2. Then $\epsilon_1 + \epsilon_2$ transforms like $triv \otimes triv$ under $W(M_1)$ and $\epsilon_1 - \epsilon_2$ transforms like $sgn \otimes triv$. The vector ϵ_1 is fixed under $W(B_n)$ (which corresponds to M_2) and the vector ϵ_2 is fixed under $W(B_{n-1})$ and transforms like $\mu_o(1,n)$ under $W(B_n)$. The matrix r_σ is, according to (6.4.7),

$$\begin{bmatrix} \frac{1}{2+\nu-c-n} & \frac{\nu-n+1-c}{2+\nu-n-c} \\ \frac{\nu-n+1-c}{1+\nu-n+1-c} & \frac{1}{2+\nu-n-c} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \frac{c+n-1-\nu}{c+n-1+\nu} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{c+\nu+n} & \frac{\nu+c+n-1}{c+\nu+n} \\ \frac{\nu+n-1+c}{c+\nu+n} & \frac{1}{c+1+\nu+n} \end{bmatrix}. \qquad (6.4.9)$$

So the vector ϵ_1 is mapped into $\frac{c+n-\nu}{c+n+\nu}\epsilon_1$ as claimed. The calculation for type D is analogous.

For μ_o we apply a similar method. Let P' be the parabolic subgroup with Levi component $GL(1)\times GL(n)$. In type B the intertwining operator has a decomposition analogous to (6.4.6). In this case the operator I_2 is the identity because in the representation μ_o the element t_n corresponding to the short simple root acts by 1. For type D let

$$\alpha_i := \epsilon_i - \epsilon_{i+1}, \quad 1 \le i \le n, \qquad \alpha_n' = \epsilon_n + \epsilon_{n+1}.$$

Let $GL(n+1)$ have simple roots $\alpha_i, i \le n$ and denote by $GL'(n+1)$ the Levi component with simple roots $\alpha_i,\ i \le n-1$ and α_n'. The intertwining operator analogous to (6.4.6) decomposes into $I_1 \circ I_1'$. Both operators are induced from Levi components of type A and so the result of the proposition is a consequence of section (6.3) and the fact that the K-types occur with multiplicity one. □

6.5. In this section we consider (6.3.2) for $k>1$, $n \ge 1$ and the K-types $\mu_e(m, n+k-m)$ for $0 \le m \le k$ which occur with multiplicity 1.

Proposition. *Assume $\sigma = \mu_e(m, n+k-m)$. In type B, $r_\sigma(\nu)$ equals*

$$\prod_{0\le j\le m-1} \frac{n+c-(-\frac{k-1}{2}+\nu)-j}{n+c+(\frac{k-1}{2}+\nu)-j} \tag{6.5.1}$$

In type D, $r_\sigma(\nu)$ equals

$$\prod_{0\le j\le m-1} \frac{n-(-\frac{k-1}{2}+\nu)-j}{n+(\frac{k-1}{2}+\nu)-j} \tag{6.5.2}$$

Proof. The case $k=1$ was done in section 6.4 so we only need to do the induction step. We factor the intertwining operator as follows. The module $X_P(\nu)$ is contained in the induced module

$$Y(\nu) = Ind_Q[\chi_{\nu'} \otimes \chi_{\frac{k-1}{2}+\nu} \otimes triv]$$

where Q has Levi component $GL(k-1)\times GL(1)\times G(n)$. This corresponds to breaking up the parameter into strings

$$((-\frac{k-1}{2}+\nu,\dots,\frac{k-3}{2}+\nu)(\frac{k-1}{2}+\nu)(-n+1-c+\epsilon,\dots,-c+\epsilon) \tag{6.5.3}$$

The intertwining operator factors

$$I = I_2' \circ I_{12} \circ I_2 \tag{6.5.4}$$

where

- I_2 changes $(\frac{k-1}{2}+\nu)$ to $(-\frac{k-1}{2}-\nu)$ and is induced from the corresponding operator coming from $GL(1)\times G(n)\subset G(n+1)$,
- I_{12} interchanges $(-\frac{k-1}{2}+\nu,\dots,\frac{k-3}{2}+\nu)$ with $(-\frac{k-1}{2}-\nu)$ and is induced from $GL(k-1)\times GL(1)\subset GL(k)$,
- I_2' changes $(-\frac{k-1}{2}+\nu,\dots,\frac{k-3}{2}+\nu)$ to $(-\frac{k-3}{2}-\nu,\dots,\frac{k-1}{2}-\nu)$ and is induced from $GL(k-1)\times G(n)\subset G(k+n)$.

The K-types that matter in the restrictions are

$$\begin{aligned}
&triv\otimes\mu_e(1,n+k-1)+triv\otimes\mu_e(0,n+k)\\
&\text{ for } GL(1)\times G(n+k-1), &(6.5.5)\\
&(k)\otimes triv+(1,k-1)\otimes triv\\
&\text{ for } GL(k)\times G(n), &(6.5.6)\\
&triv\otimes\mu_e(m-1,n+k-m)+triv\otimes\mu_e(m,n+k-1-m)\\
&\text{ for } GL(k-1)\times G(n+1). &(6.5.7)
\end{aligned}$$

The K-type $\mu_e(m,n+k-m)\cong\Lambda^m\mu_e(1,n+k-1)$. It occurs with multiplicity 2 in Y for $0<m<\min(k,n)$ and multiplicity 1 for $k=\min(k,n)$. We will write out an explicit basis for the invariant $S_1\times S_{k-1}\times W(B_n)$ vectors. The intertwining operators $I_2,\ I_2'$ in (6.5.4) are known by induction and I_{12} is computed in lemma 6.2. Then formula (6.5.2) comes down to a computation with 2×2 matrices as before. Let

$$e := \frac{1}{m!(k-m)!} \sum_{\sigma \in S_k} \sigma \cdot [\epsilon_1 \wedge \cdots \wedge \epsilon_m]. \tag{6.5.8}$$

This is the $S_k \times W(B_n)$ fixed vector of $\mu_e(m, n+k-m)$. It decomposes as

$$e = e_0 + e_1 = f_0 + f_1 \tag{6.5.9}$$

where

$$\begin{aligned}
e_0 &= \frac{1}{m!(k-1-m)!} \sum_{\sigma \in S_{k-1} \times S_1} \sigma \cdot [\epsilon_1 \wedge \cdots \wedge \epsilon_m],\\
e_1 &= \frac{1}{(m-1)!(k-m)!} \sum_{\sigma \in S_{k-1} \times S_1} \sigma \cdot [\epsilon_1 \wedge \cdots \wedge \epsilon_{m-1}] \wedge \epsilon_k,\\
f_0 &= \frac{1}{m!(k-1-m)!} \sum_{\sigma \in S_1 \times S_{k-1}} \sigma \cdot [\epsilon_2 \wedge \cdots \wedge \epsilon_{m+1}],\\
f_1 &= \frac{1}{(m-1)!(k-m)!} \sum_{\sigma \in S_1 \times S_{k-1}} \epsilon_1 \wedge \sigma \cdot [\epsilon_2 \wedge \cdots \wedge \epsilon_m].
\end{aligned} \tag{6.5.10}$$

Let also

$$\begin{aligned}
e_0' = e_0'' &= \frac{1}{(m-1)!(k-m)!} \sum_{\sigma \in S_k} \sigma \cdot [\epsilon_1 \wedge \cdots \wedge \epsilon_m],\\
e_1' &= \sum_{\sigma \in S_{k-1} \times S_1} \sigma \cdot [\epsilon_1 \wedge \cdots \wedge \epsilon_{m-1} \wedge (\epsilon_m - \epsilon_k)],\\
e_1'' &= \sum_{\sigma \in S_1 \times S_{k-1}} \sigma \cdot [(-\epsilon_1 + \epsilon_{m+1}) \wedge \epsilon_2 \wedge \cdots \wedge \epsilon_{m+1}].
\end{aligned} \tag{6.5.11}$$

Then

$$\begin{aligned}
e_0 &= \frac{k-m}{k} e_0' + \frac{m}{k} e_1', & e_1 &= \frac{m}{k} e_0' - \frac{m}{k} e_1',\\
e_0'' &= f_0 + f_1, & e_1'' &= f_0 - \frac{k-m}{m} f_1.
\end{aligned} \tag{6.5.12}$$

We now compute the action of the intertwining operators. The following relations hold:

$$
\begin{aligned}
&I_2(e_0) = e_0, \qquad I_2(e_1) = \frac{n+c-(\frac{k-1}{2}+\nu)}{n+c+(\frac{k-1}{2}+\nu)}e_1,\\
&I_{12}(e_0') = e_0'', \qquad I_{12}(e_1') = \frac{2\nu-1}{2\nu+k-1}e_1'',\\
&I_2'(f_0) = \prod_{0\le j\le m-2}\frac{n+c-(-\frac{k-1}{2}+\nu)-j}{n+c+(\frac{k-3}{2}+\nu)-j}f_0,\\
&I_2'(f_1) = \prod_{0\le j\le m-1}\frac{n+c-(-\frac{k-1}{2}+\nu)-j}{n+c+(\frac{k-3}{2}+\nu)-j}f_1
\end{aligned}
\tag{6.5.13}
$$

Then

$$
I_2(e_0+e_1) = e_0 + \frac{n+c-(\frac{k-1}{2}+\nu)}{n+c+(\frac{k-1}{2}+\nu)}e_1. \tag{6.5.14}
$$

Substituting $e_0',\ e_1'$, we get

$$
[\frac{k-m}{k} + \frac{m}{k}\frac{n+c-(\frac{k-1}{2}+\nu)}{n+c+(\frac{k-1}{2}+\nu)}]e_0' + \frac{m}{k}[1-\frac{n+c-(\frac{k-1}{2}+\nu)}{n+c+(\frac{k-1}{2}+\nu)}]e_1'. \tag{6.5.15}
$$

Applying I_2 to this has the effect that e_0' is sent to e_0'' and the term in e_1' is multiplied by $\frac{2\nu-1}{2\nu+k-1}$ and e_1' is replaced by e_1''. Substituting the formulas for e_0'' and e_1'' in terms of $f_0,\ f_1$, and applying I_2', we get the claim of the proposition. □

6.6. We now treat the case $\sigma = \mu_o(m, n+k-m)$. We assume $n>0$ or else these K-types do not occur in the induced module.

Proposition. *For type B, r_σ equals*

$$
\prod_{0\le j\le m-1}\frac{(\nu-\frac{k-1}{2})-(1-c)+j}{(\nu+\frac{k-1}{2})-(-n-c)-j}\cdot\frac{(-n-c)-(-\nu+\frac{k-1}{2})+j}{(1-c)-(-\nu-\frac{k-1}{2})-j} \tag{6.6.1}
$$

For type D, r_σ equals

$$
\prod_{0\le j\le m-1}\frac{(\nu-\frac{k-1}{2})-(1)+j}{(\nu+\frac{k-1}{2})-(-n)-j}\cdot\frac{(-n)-(-\nu+\frac{k-1}{2})+j}{(1)-(-\nu-\frac{k-1}{2})-j} \tag{6.6.2}
$$

Proof. Consider type B. We decompose the intertwining operator $I(\nu)$ that takes $X_P(\nu)$ in (6.3.1) to $X_P(-\nu)$ into

$$
I_1'\circ I_2\circ I_1 \tag{6.6.3}
$$

where

- I_1 interchanges $(-\frac{k-1}{2}+\nu,\dots\frac{k-1}{2}+\nu)$ with $(-n+1-c+\epsilon,\dots,-c+\epsilon)$ and so is induced from the corresponding operator on $GL(n+k)$,
- I_2 interchanges ν to $-\nu$ in $(-\frac{k-1}{2}+\nu,\dots\frac{k-1}{2}+\nu)$, so is induced from the corresponding operator on $G(n)$,

- I'_1 interchanges $(-n+1-c+\epsilon,\dots,-c+\epsilon)$ with $(-\frac{k-1}{2}+\nu,\dots\frac{k-1}{2}+\nu)$ and so is induced from the corresponding operator on $GL(n+k)$.

The only K-type of the form $triv\otimes\mu$ that occurs in the restriction of μ_o to $W(A_{k-1})\times W(B_n)$ is $triv\otimes triv$. Thus the operator I_2 is the identity. Then the result follows from section 6.3.

Now consider type D. If k is even the argument for type B carries over word for word. If k is odd, the same decomposition as (6.6.3) holds except that I_2 is replaced by the operator that interchanges the strings

$$(0)(-\frac{k-1}{2}+\nu,\dots,\frac{k-1}{2}+\nu)\ \mapsto\ (0)(-\frac{k-1}{2}-\nu,\dots,\frac{k-1}{2}-\nu)\quad(6.6.4)$$

and fixes all other entries. It is enough to check that the operator (6.6.4) is the identity on any μ_o. It is induced from an operator on $G(k+1)$ and decomposes further into $I_2=I'_{12}\circ I'_2\circ I'\circ I_{12}$ where

- I' changes $(-\frac{k-1}{2}+\nu,\dots,\frac{k-1}{2}+\nu)$ to $(-\frac{k-1}{2}-\nu,\dots\frac{k-3}{2}-\nu,\frac{k-1}{2}+\nu)$ and changes $GL(k)$ to $GL'(k)$,
- I_{12} interchanges (0) and $(-\frac{k-1}{2}+\nu,\dots,\frac{k-1}{2}+\nu)$,
- I'_2 changes $(\frac{k-1}{2}+\nu)(0)$ to $(-\frac{k-1}{2}-\nu)(0)$ and changes $GL'(k)$ to $GL(k)$,
- I'_{12} interchanges $(-\frac{k-1}{2}-\nu,\dots,\frac{k-1}{2}-\nu)$ and (0).

The operator I' is the identity on any μ_o because any $triv\otimes\mu$ occuring in its restriction to $W(A_{n-1})\times G(k+1)$ must have $\mu=triv$. Then I_2 is also the identity on any K-type μ_o because such a K-type only contains $triv\otimes\mu$ in its restriction to $GL(n-1)\times G(k+1)$ with μ of the form $\mu_o(1,k)$ or $\mu_o(0,k+1)$. The operator is the identity on $\mu_o(0,k+1)$. The K-type $\mu_o(1,k)$ is realized as the natural representation on the span of $\epsilon_i^2-\epsilon_j^2$. The fact that I_2 is the identity follows from a direct calculation. Using 6.3 we then conclude that $I'_{12}\circ I_{12}=Id$ on any μ_o. □

6.7. **Proof of theorem 5.3.** We use the results in the previous sections to prove the theorem in general. We give the details in the case of the group of type B and μ_e. Thus the Hecke algebra is type C, and $c=1$. There are no significant changes in the proof for the other cases. Recall the notation from section 2.3. Write

$$\nu=(x_{2m}-1/2,\dots,x_{2m}-1/2,\dots,1/2,\dots,1/2)$$

We factor the long intertwining operator so that

$$X(\nu)\xrightarrow{I_1}X_e(\nu)\xrightarrow{I_2}X(-\nu).\quad(6.7.1)$$

The claim will follow if the decomposition has the property that the operator I_1 is onto and I_2 is an isomorphism when restricted to the μ_e isotypic component.

The operator I_1 is a composition of several operators. First take the long intertwining operator induced from the Levi component $GL(n)$,

$$X(x_{2m}-1/2,\dots,\dots,1/2)\longrightarrow X(1/2,\dots,x_{2m}-1/2),\quad(6.7.2)$$

corresponding to the shortest Weyl group element that permutes the entries of the parameter form increasing order to decreasing order. The image is the induced from the corresponding irreducible spherical module $L(1/2,\dots,x_{2m}-1/2)$ on $GL(n)$. In turn this is induced irreducible from 1-dimensional spherical characters on a $GL(x_0)\times\dots\times GL(x_{2m})$ Levi component corresponding to the strings

$$(1/2,\dots,x_0-1/2)\dots(1/2,\dots,x_{2m}-1/2)$$

or any permutation thereof. This is well known by results of Bernstein-Zelevinski in the p-adic case, [V1] for the real case.

Compose with the intertwining operator

$$X(\dots(1/2,\dots,x_{2m}-1/2))\longrightarrow X(\dots(-x_{2m}-1/2,\dots,-1/2)), \quad (6.7.3)$$

all other entries unchanged. This intertwining operator is induced from the standard long intertwining operator on $G(x_{2m})$ which has image equal to the trivial representation. The image is an induced module from characters on $GL(x_0)\times\dots\times GL(x_{2m-1})\times G(x_{2m})$. Now compose with the intertwining operator

$$\begin{aligned}&X(\dots(1/2,\dots,x_{2m-1})(-x_{2m}-1/2,\dots,-1/2)) \qquad (6.7.4)\\ &\quad\longrightarrow X(\dots(-x_{2m-1},\dots,-1/2)(-x_{2m}-1/2,\dots,-1/2))\end{aligned}$$

(again all other entries unchanged). This is induced from the corresponding operator on $GL(x_0)\times\dots\times G(x_{2m}+x_{2m-1})$, and by section (6.2) its restriction of (6.7.4) to the μ_e isotypic component is an isomorphism. Now compose this operator with the one corresponding to

$$\begin{aligned}&X(\dots(1/2,\dots,x_{2m-2})(-x_{2m-1}+1/2,\dots,1/2)\dots) \qquad (6.7.5)\\ &\quad\longrightarrow X(\dots(-x_{2m-1},\dots,x_{2m-2}-1/2)\dots)\end{aligned}$$

with all other entries unchanged. This is induced from $GL(x_0)\times\dots\times GL(x_{2m-2}+x_{2m-1})\times G(x_{2m})$ and the image is the representation induced from the character corresponding to the string

$$(-x_{2m-1}-1/2,1/2,\dots,x_{2m-2}) \text{ on } GL(x_{2m-2}+x_{2m-1}).$$

Now compose further with the intertwining operator

$$\begin{aligned}&X(\dots(-x_{2m-1}+1/2),\dots,x_{2m-2}-1/2)(-x_{2m}-1/2,\dots,-1/2)) \quad (6.7.6)\\ &\quad\longrightarrow X((-x_{2m-1}+1/2,\dots,x_{2m-2}-1/2)\dots(-x_{2m}-1/2,\dots,-1/2))\end{aligned}$$

from the representation induced from

$$GL(x_0)\times\dots\times GL(x_{2m-3})\times GL(x_{2m-2}+x_{2m-1})\times G(x_{2m})$$

to the induced from

$$GL(x_{2m-2}+x_{2m-1})\times GL(x_0)\times\dots\times GL(x_{2m-3})\times G(x_{2m}).$$

By lemma 6.2, this intertwining operator is an isomorphism on any μ_e isotypic component. In fact, because the strings are nested, the results mentioned earlier for $GL(n)$ imply that the induced modules are isomorphic.

We have constructed a composition of intertwining operators from the standard module $X(\nu)$ where the coordinates of ν are positive and in decreasing order (*i.e.* dominant) to a module induced from

$$GL(x_{2m-2}+x_{2m-1}) \times GL(x_0) \times \cdots \times GL(x_{2m-3}) \times G(x_{2m})$$

corresponding to the strings

$$\begin{aligned}((-x_{2m-1}+1/2,\dots,x_{2m-2})(1/2,\dots,x_0-1/2),\dots\\ \dots(-x_{2m}+1/2,\dots,-1/2))\end{aligned}$$

so that the restriction to any μ_e isotypic component is onto. We can repeat the procedure with x_{2m-4}, x_{2m-3} and so on to get an intertwining operator from $X(\nu)$ to the induced from

$$GL(x_{2m-1}+x_{2m-2}) \times \cdots \times GL(x_1+x_0) \times G(x_{2m})$$

corresponding to the strings

$$\begin{aligned}((-x_{2m-1}+1/2,\dots,x_{2m-2})\dots(-x_1+1/2,\dots,x_0-1/2),\\ (-x_{2m}+1/2,\dots,-1/2)).\end{aligned}$$

Since by lemma 6.2 the intetwining operators permuting the GL-factors are isomorphisms, we get I_1 with the claimed properties.

We now deal with I_2. Consider the group $G(x_1+x_0+x_{2m})$ and P the standard parabolic subgroup with Levi component $M = GL(x_1+x_0) \times G(x_{2m})$. Let P' be the standard parabolic subgroup with Levi component $M' = GL(x_1) \times GL(x_0) \times G(x_{2m})$. Let

$$\begin{aligned}&\eta = (-x_1+1/2,\dots,x_0-1/2,-x_{2m}+1/2,\dots,-1/2),\\ &\chi_\eta \longleftrightarrow (-x_1+1/2,\dots,x_0-1/2).\end{aligned} \tag{6.7.7}$$

The induced module

$$X_P := Ind_M^G[\chi_\eta \otimes triv] \tag{6.7.8}$$

corresponding to the strings

$$(-x_1+1/2,\dots,x_0-1/2)(-x_{2m}+1/2,\dots,-1/2). \tag{6.7.9}$$

is a submodule of

$$X_{P'}(\eta',\eta'') := Ind_{M'}^G[\chi_{\eta'} \otimes \chi_{\eta''} \otimes triv] \tag{6.7.10}$$

corresponding to the strings

$$(-x_1+1/2,\dots,-1/2)(1/2,\dots,x_0-1/2)(-x_{2m}+1/2,\dots,-1/2). \tag{6.7.11}$$

which in turn is a submodule of $X(\eta)$. Now consider the intertwining operator

$$I(\eta',\eta'') \;:\; X_{P'}(\eta',\eta'') \longrightarrow X_{P'}(\eta',-\eta''). \tag{6.7.12}$$

It is an isomorphism on the μ_e isotypic components because it is induced from an intertwining operator on $G(x_0+x_{2m})$ by formula (6.5.1). Inducing up to the modules on the original group $G(n)$, we find an intertwining operator from an induced module from

$$GL(x_{2m-1}+x_{2m-2})\times\cdots\times GL(x_1+x_0)\times G(x_{2m})$$

corresponding to the string

$$\begin{aligned}(-x_{2m-1}+1/2,\dots,x_{2m-2}-1/2)\dots(-x_1+1/2,\dots\\ \dots,x_0-1/2)(-x_{2m}+1/2,\dots,1/2)\end{aligned}$$

to the module induced from

$$GL(x_{2m-1}+x_{2m-2})\times\cdots\times GL(x_1)\times GL(x_0)\times G(x_{2m})$$

corresponding to the string

$$\begin{aligned}&(-x_{2m-1}+1/2,\dots,x_{2m-2}-1/2)\dots(-x_1+1/2,\dots,-1/2)\\ &(-x_0+1/2,\dots,1/2)(-x_{2m}+1/2,\dots,1/2)\end{aligned}\tag{6.7.13}$$

which is injective on the μ_e isotypic component. Since the strings in (6.7.13) are nested, the intertwining operators that permute the GL factors are isomorphisms. So we permute $GL(x_0)$ and $GL(x_1)$ with $GL(x_3+x_4)$, and the other $GL(x_{2i-2}+x_{2i-1})$ and repeat the argument we gave for $x_0,\ x_1$ with $x_2,\ x_3$ and so on. We end up with an intertwining operator to the induced module from

$$GL(x_1)\times GL(x_0)\times\cdots\times GL(x_{2m-1})\times GL(x_{2m-2})\times G(x_{2m})\tag{6.7.14}$$

corresponding to the strings

$$\begin{aligned}&(-x_1+1/2,\dots,1/2)(-x_0+1/2),\dots,-1/2)\dots\\ &(-x_{2m-1}+1/2,\dots,-1/2)(-x_{2m-2}+1/2,\dots,-1/2)(-x_{2m}+1/2,\dots,1/2)\end{aligned}\tag{6.7.15}$$

which is injective on the μ_e isotypic components. Finally, by properties of intertwining operators on $GL(n)$ already mentioned, this last module maps injectively to $X(-\nu)$.

This completes the proof for this case. The case of μ_o is similar and we omit the details.

6.8. We now describe the modifications needed to prove corollary 5.3 in the case when the parameter is formed of half-integers (still type B and details for μ_e only) but not necessarily unipotent. The proof is no more difficult. We factor the long intertwining operator

$$X(\nu)\xrightarrow{I_1}X_e(\nu)\xrightarrow{I_2}X(-\nu)\tag{6.8.1}$$

such that I_1 is onto all the μ_e isotypic component and I_2 is into. The module X_e is defined by the strings specified in 2.5. We describe it again when we define I_1. So recall the notation for type B in section 2.5. Denote the coordinates of ν as

$$(R-1/,\dots,R-1/2,\dots,r-1/2,\dots,r-1/2),\qquad R\ge r>0. \tag{6.8.2}$$

The long intertwining operator for $GL(n)$ induces an interwining operator

$$X(\nu)\longrightarrow X(\nu') \tag{6.8.3}$$

where ν' is as in (2.4.1); the entries are the same as for ν but in increasing order. The image of this operator is the module $Ind_{GL(n)}^{G(n)}L_{GL(n)}(\nu)$. Break the parameter up into nested strings so that $r_i\le r_{i+1}\le R_{i+1}\le R_i$

$$(r_1-1/2,\dots,R_1-1/2)\dots(r_k-1/2,\dots,R_k-1/2). \tag{6.8.4}$$

Then $L_{GL}(\nu)$ is induced irreducible from the characters corresponding to these strings on $GL(R_1-r_1+1)\times\dots\times GL(R_k-r_k+1)$ and so is the image of the operator in (6.8.3). Reorder the strings so that the ones starting with 1/2's are last and otherwise they are in increasing length from left to right. Call this new parameter ν'' and label its strings

$$\begin{aligned}&(a_1-1/2,\dots,A_1-1/2)\dots(a_l-1/2,\dots,A_l-1/2)\\&(1/2,\dots x_0-1/2)\dots(1/2,\dots,x_{2m}-1/2).\end{aligned} \tag{6.8.5}$$

Let

$$M''=GL(A_1-a_1+1)\times\dots GL(A_l-a_l+1)\times GL(x_0)\times\dots GL(x_{2m}),$$

and denote by η_i the character corresponding to the string $(a_i-1/2,\dots,A_i-1/2)$ and by χ_j the character corresponding to $(1/2,\dots,x_i-1/2)$. Composing (6.8.3) with intertwining operators permuting the GL factors we get an intertwining operator

$$X(\nu)\longrightarrow Ind_{M''}^{G}[\eta_1\otimes\dots\otimes\eta_l\otimes\chi_0\otimes\dots\otimes\chi_{2m}] \tag{6.8.6}$$

which is onto when restricted to the μ_e isotypic components. Let

$$\begin{aligned}M_e:=&GL(A_1-a_1+1)\times\dots\times GL(A_l-a_l+1)\times\\&GL(x_{2m-1}+x_{2m-2})\times\dots\times GL(x_1+x_0)\times G(x_{2m}),\end{aligned} \tag{6.8.7}$$

and let $X_e(\nu)$ be the induced module from M_e corresponding to the strings

$$\begin{aligned}&(a_1-1/2,\dots,A_1-1/2)\dots(a_l-1/2,\dots,A_l-1/2)\\&(-x_{2m-1}+1/2,\dots,x_{2m-2}-1/2)\dots\\&(-x_1+/12,\dots,x_0-1/2)(-x_{2m}+1/2,\dots,-1/2).\end{aligned} \tag{6.8.8}$$

The proof of theorem 5.3 gives an operator

$$Ind_{M''}[\eta_1\otimes\dots\otimes\eta_l\otimes\chi_0\otimes\dots\otimes\chi_{2m}]\longrightarrow X_e(\nu) \tag{6.8.9}$$

which is onto when restricted to the μ_e isotypic component. The composition of (6.8.6) with (6.8.9) is the operator

$$I_1:X(\nu)\longrightarrow X_e(\nu). \tag{6.8.10}$$

We now describe I_2. The operator from $X_e(\nu)$ to the induced module from

$$M_e' := GL(A_1 - a_1 + 1) \times \cdots \times GL(A_{l-1} - a_{l-1} + 1) \times \\ GL(x_{2m-1} + x_{2m-2}) \times \cdots \times GL(x_1 + x_0) \times GL(A_l - a_l + 1) \times G(x_{2m}) \tag{6.8.11}$$

which permutes the GL factors and the strings is an isomorphism on the μ_e isotypic components by lemma 6.2. The intertwining operator that changes the string

$$(a_l - 1/2, \dots, A_l - 1/2) \quad \text{to} \quad (-A_l + 1/2, \dots, -a_l - 1/2) \tag{6.8.12}$$

is an isomorphism on the μ_e isotypic component by formula (6.5.1). This uses the fact that $a_l > 1$. Then any intertwining operator that interchanges GL factors and their strings is an isomorphism on the μ_e component by lemma 5.3. Let $X_e(-\nu)$ be the module induced from M_e where all the strings on the GL's are reversed as in (6.8.12). We have constructed an intertwining operator

$$X_e(\nu) \longrightarrow X_e(-\nu) \tag{6.8.13}$$

which is an isomorphim on the μ_e isotypic components. The fact that the operator

$$X_e(-\nu) \longrightarrow X(-\nu) \tag{6.8.14}$$

is an isomorphism on the μ_e isotypic component is a consequence of properties of the intertwining operators on groups of type A. The operator I_2 is the composition of (6.8.13) and (6.8.14). The proof follows.

7. The Induction

To check that condition [B] is necessary, we will do an induction on the rank of the Lie algebra $\mathfrak{g}$ and downward on the nilpotent orbit $\check{\mathcal{O}}$ (attached to the parameter) ordered by inclusion in closures. **Assume** $c = 1$.

7.1. Consider the representation corresponding to

$$(a - \epsilon + \nu, \dots, A - \epsilon + \nu)(-n + \epsilon, \dots, -1 + \epsilon), \quad |a| \le A, \ 0 < \nu < 1, \tag{7.1.1}$$

where $a, A \in \mathbb{Z}$, $\epsilon = 1/2$ for type B, $\epsilon = 0$ for type C and $\epsilon = 1$ for type D. The second string represents the infinitesimal character corresponding to the principal nilpotent $\check{\mathcal{O}}_0$.

Proposition. *The form on $L(\chi)$ corresponding to (7.1.1) is negative on the following K-type:*

1. *If $n < a$ then the form is negative on $(n + A - a) \times (1)$.*
2. *If $a \le n < A$ then the form is negative on $(A - 1) \times (n - a + 2)$.*
3. *If $A \le n$ and $a - 2\epsilon \ge 0$ then the form is negative on $(1, n + A - a) \times (0)$.*
4. *If $A \le n$ and $a - 2\epsilon < 0$ then the form is negative on $(-a + 1, n + A) \times (0)$.*

Proof. This is a corollary of the results in section 6.2. □

We will say a spherical irreducible module is *a-unitary* if the form is positive on the K-types μ_e and μ_o. Similarly, for an induced module, *a-irreducible* means that all K-types of the form μ_e K-types occur with the same multiplicity in X_e as in $L(\chi)$ or that all K-types of the form μ_o occur with full mulitplicity in X_o and $L(\chi)$.

7.2. **Initial Step.** Consider the case of a parameter corresponding to a $\check{\mathcal{O}}$ which has a unipotent part corresponding to an even nilpotent orbit $\check{\mathcal{O}}_0$ and a single string. We write the string as

$$(a-\epsilon,\dots,A-\epsilon)+\nu(1,\dots,1), \qquad |a|\le A,\ 0\le\nu<1. \tag{7.2.1}$$

We do the case of type C only, the others are similar. The nilpotent orbit $\check{\mathcal{O}}_0$ corresponds to the partition $(2x_0+1,\dots,2x_{2m}+1)$ and the parameter has strings

$$(1,\dots x_0)(0,1,\dots,x_1)\dots(1,\dots,x_{2m}).$$

The partition of $\check{\mathcal{O}}$ is $(2x_0+1,\dots,2x_{2m}+1,A-a+1,A-a+1)$. We may as well deform ν so that $\nu=1/2$, Since no reducibility occurs, the signatures of all K-types stay unchanged. We want to show that if $A+a>0$, or if $A+a=0$ and there is no $x_i=A$, then $L(\chi)$ is *not* unitary. Since we do a downward induction on the rank of $\mathfrak{g}$ and downward on $\check{\mathcal{O}}$ the first case is when $\check{\mathcal{O}}$ is maximal. This is the principal nilpotent (so $m=0$),and the claim follows from proposition 7.1. So we assume that m is strictly greater than 0.

Assume $\mathbf{x_{2i}<A\le x_{2i+1}}$. We will show that the form is negative on a K-type $\mu_e(k)$. If there is any pair $x_{2j}=x_{2j+1}$, the module X_e is unitarily induced from $G(n-2x_{2j}-1)\times GL(2x_{2j}+1)$ and all K-types $\mu_e(k)$ have the same multiplicity in $L(\chi)$ as in X_e. We can *remove* the string corresponding to $(x_{2j}x_{2j+1})$ in X_e as explained in section 3. By induction on rank we are done. Similarly we can remove any pair (x_{2j},x_{2j+1}) such that either $x_{2j+1}\le|a|$ or $A\le x_{2j}$ as follows. Deform the string

$$(-x_{2j+1},\dots,x_{2j}) \text{ to } (-\frac{x_{2j}+x_{2j+1}+1}{2},\dots,\frac{x_{2j}+x_{2j+1}+1}{2}). \tag{7.2.2}$$

No a-reduciblility occurs, so the multiplicities and signatures in X_e and $L(\chi)$ do not change. The new X_e is unitarily induced from $triv\otimes X_e'$ on $GL(x_{2j}+x_{2j+1})\times G(n-x_{2j}-x_{2j+1}-1)$ and we can *remove* the string corresponding to $(x_{2j}x_{2j+1})$. The induction hypothesis applies to X_e'.

When $A+a=0$, the above argument implies that it is enough to consider the case

$$\check{\mathcal{O}}\longleftrightarrow(x_0,x_1,x_2), \qquad x_0<M<x_1\le x_2. \tag{7.2.3}$$

We reduce to (7.2.3) when $A+a>0$ as well. By the above arguments, we may as well assume $m=2i+2$. Suppose there is a pair (x_{2j},x_{2j+1}) such that $|a|<x_{2j+1}$, and $j\ne i$. The assumption is that $x_{2i}<A\le x_{2i+1}$

so $x_{2j+1} \le x_{2i} < A$. We can deform the character in the parameter of X_e corresponding to (x_{2j}, x_{2j+1}) to

$$(-x_{2j+1}+\nu,\dots,x_{2j}+\nu) \qquad \text{or} \qquad (-x_{2j}+\nu,\dots,x_{2j+1}+\nu). \tag{7.2.4}$$

The multiplicities of the $\mu_e(k)$ in X_e and $L(\chi)$ are equal until the parameter reaches ν. So if the signature on some $\mu_e(k)$ isotypic component is indefinite, the signature has to be indefinite on the original $L(\chi)$. But the induction hypothesis applies to at least one of these parameters, and implies that the form is indefinite on a $\mu_e(k)$. For example, if $a<0$ use the first deformation. The strings for the new $L(\chi)$ are

$$(-x_{2j+1}+\nu,\dots,A+\nu) \qquad (a+\nu,\dots,x_{2j}+\nu).$$

Then $A + x_{2j+1} > 0$. Deform ν in the second string to zero. The new nilpotent has $\check{\mathcal{O}}'$ with partition

$$(\dots,2|a|+1,\dots,\widehat{2x_{2j+1}+1},\dots,A+x_{2j+1}+1,A+x_{2j+1}+1,\dots)$$

which contains $\check{\mathcal{O}}$ in its closure. By induction the form is indefinite on a K-type $\mu_e(k)$.

We have reduced to case (7.2.3). We now reduce further to the case

$$\check{\mathcal{O}}_0 \longleftrightarrow (x_0), \qquad x_0 < A. \tag{7.2.5}$$

which is the initial step. Consider the module $I(\nu')$ for $0 \le \nu' < 1$ corresponding to the strings

$$(-x_1+\nu',\dots,x_2+\nu')(a+\nu,\dots,A+\nu)(-x_0,\dots,-1). \tag{7.2.6}$$

i.e. induced from

$$GL(a+A+1)\times GL(x_1+x_2)\times G(x_0). \tag{7.2.7}$$

Consider the irreducible spherical module in the induced from just $GL(a+A+1)\times G(x_0)$. By section 7.1, the form is, negative on $\mu_e(1)$ if $x_0 < a$, negative on $\mu_e(x_0+1-a)$ if $a \le x_0$. In the second case the form is positive on all $\mu_e(x_0+1-a-r)$ for $1<r<n+1-a$. So let $r_0 := 1$ or x_0+1-a depending on these two cases. The multiplicity formulas from section 6.2 imply that

$$[\mu_e(r_0) \;:\; I(\nu')] = [\mu_e(r_0) \;:\; L(\chi)] \quad \text{for} \quad 0 \le \nu' \le \frac{x_2-x_1}{2}.$$

Thus we can deform ν' to $\frac{x_2-x_1}{2}$, where $I(\nu')$ is unitarily induced, and conclude that the form on $L(\chi)$ is negative on $\mu_e(r_0)$.

Assume $\mathbf{x_{2i-1} < A \le x_{2i}}$. In this case we can do the same arguments using X_o and μ_o. We omit the details.

7.3. **Induction step.** We will show that an a-unitary spherical module has to satisfy property (B) by induction on the rank of $\mathfrak{g}$ and downward on the nilpotent orbits ordered by inclusions in closures in the dual group. We treat the case of the group of type C only.

The case of a single string was done in section 6.2; so assume there is more than one string. The spherical representation corresponding to ν is induced irreducible from a

$$L_0 \otimes \chi_{\nu_1} \otimes \cdots \otimes \chi_{\nu_l}$$

where L_0 is special unipotent. There are at least two strings, label them

$$(f+\nu_1, \dots, F+\nu_1), \qquad (e+\nu_2, \dots, E+\nu_2). \tag{7.3.1}$$

Suppose the first string does not satisfy (B). This means that $F+f>0$ or $F+f<-2$. We treat the case $F+f<-2$ (so $f \leq -2$ because of the convention that $|f| \leq F$). The other case is similar.

The strategy is to deform ν_2 until the first time the induced module becomes reducible. The nilpotent orbit attached to the spherical subquotient is larger so the induction hypothesis applies to the spherical subquotient. Assume the reducibility occurs at a $\nu_2 \in \mathbb{Z}$. There are several possibilities. Supopse the string corresponding to ν_2 combines with the parameter of L_0 to form a new parameter corresponding to a strictly larger nilpotent. The string corresponding to ν_1 stays unchanged, and does not satisfy condition (B). This contradicts the induction hypothesis. The same argument applies if no a-reducibility occurs while ν_2 can be deformed to a value where the parameter is unitarily induced. We remove a string and the induction hypothesis applies because the rank of the algebra is strictly lower.

Suppose ν_2 can be deformed to ∞ without a-reducibility occuring. Then we can deform ν_2 to be very large compared to all other coordinates, and deform one of the remaining strings of X_e so that the parameter becomes unitarily induced. The induction hypothesis implies that the form is negative on $\mu_e(1)$.

Assume that the reducibility occurs at a $\boldsymbol{\nu_2} \notin \mathbb{Z}$. When the representation becomes reducible, the string corresponding to ν_2 combines with another string to give a parameter corresponding to a strictly larger nilpotent orbit. If the string corresponding to ν_1 is not involved, we get a contradiction to the induction hypothesis. We are reduced to the case when the reducibility involves the first string in (7.3.1). We can deform ν_2 in two directions. One of them gives reducibility when ν_2 is deformed to ν_1. The strings of the new $L(\chi)$ are

$$\begin{aligned} &(f+\nu_1, \dots, E+\nu_1),\ (e+\nu_1, \dots, F+\nu_1) \text{ if } f<e \leq F<E, \\ &(f+\nu_1, \dots, E+\nu_1) \text{ if } F=e-1. \end{aligned} \tag{7.3.2}$$

(There is also the case $E = f - 1$, but then $e + E < -2$ as well and we can interchange the labeling of the strings). In both cases the induction hypothesis implies $-2 \leq f + E \leq 0$ (so $E \geq 0$). We show that $e \leq 0$. In the first case $-2 \leq e + F \leq 0$ as well, so $e \leq 0$. If in the second case $e > 0$, then $F = e - 1 \geq 0$, and the second string does not satisfy (B) either. If $\nu_1 < \nu_2$ as well, then deform $\nu_1 \searrow 0$. The induced representations has to become reducible and the reducibility does not involve the string with ν_2. But since $e > 0$ the second string in (7.3.1) does not satisfy (B), contradicting the induction hypothesis. Thus consider the case $\nu_2 < \nu_1$. Deform $\nu_1 \nearrow 1$. The first reducibility point has to be at $\nu_1 = 1 - \nu_2$. The strings of the new $L(\chi)$ are

$$(e + \nu_2, \dots, E + \nu_2), \quad (-e + \nu_2, \dots, -f - 1 + \nu_2). \tag{7.3.3}$$

Our assumptions are $f + F = f + e - 1 < -2$, so the induction hypothesis implies $-2 \leq -e + E \leq 0$ and $-2 \leq E - f - 1 \leq 0$, a contradiction.

Thus $\mathbf{e} \leq \mathbf{0}$. If $\nu_2 < \nu_1$, deform $\nu_2 \searrow 0$. The resulting parameter does not satisfy the induction hypothesis. So $\mathbf{e} \leq \mathbf{0}$, $\boldsymbol{\nu_1} < \boldsymbol{\nu_2}$ must hold in all cases. Suppose $e = 0$. If $e \leq F$ then move $\nu_1 \searrow 0$. The induction hypothesis implies $-2 \leq e + E \leq 0$ so $E = 0$. But in this case $e \leq F < E$, a contradiction. If on the other hand $F = e - 1 = -1$, the strings are

$$(f + \nu_1, \dots, -1 + \nu_1), \qquad (0 + \nu_2, \dots, E + \nu_2). \tag{7.3.4}$$

Deform ν_2 upward. The induced module cannot be reducible at $\nu_2 = 1 - \nu_1$ or at $\nu_2 = 1$. If it were reducible, the induction hypothesis applies but the string with ν_1 cannot be involved and does not satisfy (B). The same argument applies at the next possible reducibility point $\nu_2 = 1 + \nu_1$; the strings are

$$(f + \nu_1, \dots, -1 + \nu_1), \qquad (1 + \nu_1, \dots, E + \nu_1). \tag{7.3.5}$$

We conclude that we can deform ν_2 to ∞ and no reducibility can occur. So make ν_2 very large. Then deform $\nu_1 \nearrow 1$ in the first string so that the resulting spherical module is earlier in the induction. This contradicts the induction hypothesis since the string with ν_2 does not satisfy (B).

Thus we are reduced to the case when we may assume that $\mathbf{e} < \mathbf{0}$, $\boldsymbol{\nu_1} < \boldsymbol{\nu_2}$ and the induced module has to be reducible at $\nu_2 = 1 - \nu_1$. The strings of $L(\chi)$ become

$$(f + \nu_1, \dots, F + \nu_1) \qquad (-E - 1 + \nu_1, \dots, -e - 1 + \nu_1). \tag{7.3.6}$$

The induction hypothesis implies

$$\begin{aligned} &-2 \leq f - e - 1 \leq 0, \qquad -2 \leq F - E - 1 \leq 0, \text{ or} \\ &-2 \leq f - e - 1 \leq 0, \qquad F = -E - 2. \end{aligned} \tag{7.3.7}$$

In the first case we get $f = -2, F = -1$ and $e = -1, E = 0$. Deforming $\nu_2 \nearrow 1/2$ we find a unitarily induced a-irreducible module from a similar one on a proper Levi component which contradicts the induction hypothesis. In the second case we find $f = F = -E - 2$. The same argument applies.

8. Real nilpotent orbits

In this section we review some well known results for real nilpotent orbits. Some additional details and references can be found in [CM].

8.1. Fix a real form $\mathfrak{g}_0$ of a complex semisimple Lie algebra $\mathfrak{g}$. Let θ be the complexification of the Cartan involution of $\mathfrak{g}_0$, and write $^-$ for the conjugation. Let G be the adjoint group with Lie algebra $\mathfrak{g}$, and let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{s}, \qquad \mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0 \tag{8.1.1}$$

be the Cartan decomposition. Write $K \subset G$ for the subgroup corresponding to $\mathfrak{k}$, and G_0 and K_0 for the real Lie groups corresponding to $\mathfrak{g}_0$ and $\mathfrak{k}_0$.

Let $e \in \mathfrak{g}$ be a nilpotent element.

Theorem (Jacobson-Morozov). (1) *There is a one to one correspondence between G-orbits of nilpotent elements and G-orbits of* Lie triples $\{e,\ h,\ f\}$ *i.e. elements satisfying*

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

This correspondence is realized by completing every nilpotent element e to a Lie triple.

(2) *Two Lie triples $\{e,\ h,\ f\}$ and $\{e',\ h',\ f'\}$ are conjugate if and only if the elements h and h' are conjugate.*

8.2. Suppose $e \in \mathfrak{g}_0$ is nilpotent. Then one can still complete it to a Lie triple $e,\ h,\ f \in \mathfrak{g}_0$. Such a Lie triple is called *real* or ρ stable. A Lie triple is called *Cayley* if in addition $\theta(h) = -h,\ \theta(e) = f$. Every real Lie triple is conjugate to one which is Cayley. Theorem 8.1 is no longer true, but the following modification holds.

Theorem (Kostant-Rao). *Two real Lie triples are conjugate if and only if the elements $e - f$ and $e' - f'$ are conjugate under G_0. Equivalently, two Cayley triples are conjugate if and only if $e - f$ and $e' - f'$ are conjugate under K_0.*

8.3. Suppose $e \in \mathfrak{s}$ is nilpotent. Then e can be completed to a Lie triple satisfying

$$\theta(e) = -e, \qquad \theta(h) = h, \qquad \theta(f) = -f. \tag{8.3.1}$$

We call such a triple θ-stable. To any Cayley triple one can associate a θ-stable triple as in (8.3.1), by the formulas

$$\widetilde{e} := \frac{1}{2}(e + f + ih), \quad \widetilde{h} := i(e - f), \quad \widetilde{f} := \frac{1}{2}(e + f - ih). \tag{8.3.2}$$

A Lie triple is called *normal* if in addition to (8.3.1) it satisfies $\overline{e} = f,\ \overline{h} = -h$.

Theorem (Kostant-Sekiguchi). (1) *Any θ-stable triple is conjugate via K to a normal one.*

(2) *Two nilpotent elements $\widetilde{e}$, $\widetilde{e}' \in \mathfrak{s}$ are conjugate by K, if and only if the corresponding Lie triples are conjugate by K. Two θ-stable triples are conjugate under K if and only if the elements $\widetilde{h}$, $\widetilde{h}'$ are conjugate under K.*

(3) *The correspondence (8.3.2) is a bijection between G_0 orbits of nilpotent elements in $\mathfrak{g}_0$ and K orbits of nilpotent elements in $\mathfrak{s}$.*

Proposition. *The correspondence between real and θ stable orbits is compatible with closure relations.*

Proof. This is the main result in [BS]. □

8.4. Let $\mathfrak{p}_0 = \mathfrak{m}_0 + \mathfrak{n}_0$ be a real parabolic subalgebra and $e \in \mathfrak{m}_0$ be a nilpotent element.

Definition. *The ρ-induced set from e to $\mathfrak{g}_0$ is the finite union of orbits $\mathcal{O}(E_i) := \operatorname{Ad} G_0 e_i$ such that*

$$\text{each } \mathcal{O}(E_i) \text{ is open in } \operatorname{Ad} G_0(e + \mathfrak{n}_0) \text{ and } \overline{\bigcup \mathcal{O}(E_i)} = \operatorname{Ad} G_0(e + \mathfrak{n}_0).$$

We write

$$ind_{\mathfrak{p}_0}^{\mathfrak{g}_0}(\mathcal{O}_{\mathfrak{m}_0}(e)) = \bigcup \mathcal{O}(E_i). \tag{8.4.1}$$

and we say that each E_i is real or ρ induced from e.

The ρ-induced set depends on e and the Levi component $\mathfrak{m}_0$, but not on $\mathfrak{n}_0$. In terms of the θ-stable versions $\widetilde{e}$ of e, and $\widetilde{E}_i$ of E_i, ρ-induction is computed in [BB]. This is as follows. Let $\mathfrak{h} \subset \mathfrak{m}$ be a maximally split real Cartan subalgebra, and $\xi \in \mathcal{Z}(\mathfrak{m}_0) \cap \mathfrak{s}$ an element of $\mathfrak{h}_0$ such that

$$\alpha \in \Delta(\mathfrak{n}_0, \mathfrak{h}_0) \text{ if and only if } \alpha(\xi) > 0.$$

Then

$$\overline{\bigcup \mathcal{O}_K(\widetilde{E}_i)} = \overline{\bigcup_{t>0} \operatorname{Ad} K(t\xi + \widetilde{e})} \setminus \bigcup_{t>0} \operatorname{Ad} K(t\xi + \widetilde{e}). \tag{8.4.2}$$

8.5. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ-stable parabolic subgroup, and write $\overline{\mathfrak{q}} = \mathfrak{l} + \overline{\mathfrak{u}}$ for its complex conjugate. Let $e \in \mathfrak{l} \cap \mathfrak{s}$ be a nilpotent element.

Proposition. *There is a unique K-orbit orbit $\mathcal{O}_K(E)$ so that its intersection with $\mathcal{O}_{L\cap K}(e) + (\mathfrak{u} \cap \mathfrak{s})$ is open and dense.*

Proof. This follows from the fact that $e + (\mathfrak{u} \cap \mathfrak{s})$ is formed of nilpotent orbits, there are a finite number of nilpotent orbits, and being complex, the K-orbits have even real dimension. □

Definition. *The orbit $\mathcal{O}_K(E)$ as in the proposition above is called θ-induced from e, and we write*

$$ind_{\mathfrak{q}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}}(e)) = \mathcal{O}(E),$$

and say that E is θ-induced from e.

The induced orbit is characterized by the property that it is the (unique) largest dimensional one which meets $e + \mathfrak{u} \cap \mathfrak{s}$. It depends on e as well as $\mathfrak{q}$, not just e and $\mathfrak{l}$.

8.6. Consider $\mathbb{Z}_2 \rtimes sl(2,\mathbb{C})$, where the nontrivial element $\theta \in \mathbb{Z}_2$ acts on $sl(2,\mathbb{C})$ by (8.3.1). Let (π, V) be an irreducible representation of $sl(2,\mathbb{C})$ of dimension $n+1$ and let $\{v_i\}$ be a basis so that

$$\pi(e)v_i = a_i v_{i+2}, \qquad \pi(h)v_i = iv, \qquad \pi(f)v_i = v_{i-2}. \tag{8.6.1}$$

Proposition. *The representation (π, V) extends in two inequivalent ways to $\mathbb{Z}_2 \rtimes sl(2,\mathbb{C})$ according to whether θ acts by ± 1 on v_n.*

Proof. This is straightforward. □

In general, for a not necessarily irreducible (π, V), we define its *signature* to be the pair of integers (a_+, a_-), where $a_\pm$ is the dimension of the ± 1 eigenspace of θ on the kernel of $\pi(e)$.

8.7. **u(p,q).** Let V be a finite dimensional vector space of dimension n. There are two inner classes of real forms of $gl(V)$. One is such that θ is an outer automorphism. It consists of the real form $GL(n,\mathbb{R})$, and when n is even, also $U^*(n)$. The other one is such that θ is inner, and consists of the real forms $U(p,q)$ with $p+q=n$. In sections 8.7-8.13, we investigate ρ and θ induction for these forms, and then derive the corresponding results for $so(p,q)$ and $sp(n,\mathbb{R})$ from them in sections 8.14-8.15. Thus assume that V is the complexification of a real vector space V_0, and is endowed with a positive definite hermitian inner form $\langle\ ,\ \rangle$, which is symmetric when restricted to V_0. Let $\theta \in GL(V)$ be an elementof order 2. It determines a hermitian form $(v,w) := \langle \theta v, w\rangle$ on V. If θ has p eigenvalues equal to 1 and q eigenvalues equal to -1, then the hermitian form has signature (p,q). The group of transformations which are unitary for $(\ ,\ )$ is $U(p,q)$.

We need some results about closure relations between nilpotent orbits. For a θ-stable nilpotent element e, we write $a_\pm(e^k)$ for the signature of θ on the kernel of e^k, and $a(e^k) = a_+(e^k) + a_-(e^k)$ for the dimension of the kernel. If it is clear what nilpotent element they refer to, we will abbreviate them as $a_\pm(k)$.

Theorem. *Two θ stable nilpotent elements e and e' are conjugate by K if and only if e^k and e'^k have the same signatures. The relation $\mathcal{O}_K(e') \subset \overline{\mathcal{O}_K(e)}$ holds if and only if for all k,*

$$a_+(e'^k) \geq a_+(e^k), \qquad a_-(e'^k) \geq a_-(e^k).$$

Proof. This follows from [D] and proposition 8.3. We give some details which will be useful later.

Let e be a θ-stable nilpotent orbit. Decompose

$$V = \bigoplus V_i$$

into $\mathbb{Z}_2 \rtimes sl(2)$ representations and let ϵ_i be the eigenvalue of θ on the highest eigenweight of V_i. We encode the information about e into a *tableau* with rows equal to the dimensions of V_i and alternate signs $+$ and $-$ starting with the sign of ϵ_i. The number of $+$'s and $-$'s in the first column gives the signature of θ on the kernel of e. Then the number of $\pm$ in the first two columns gives the signature of θ on the kernel of e^2 and so on. The number of $+$'s equals p, the number of $-$'s equals q. Write $V = V_+ + V_-$, where $V_\pm$ are the ± 1 eigenspaces of θ. The element e is given by a pair (A, B), where $A \in \mathrm{Hom}[V_+, V_-]$, and $B \in \mathrm{Hom}[V_-, V_+]$. Then e^k is represented by $(ABAB\ldots, BABA\ldots)$, and $a_\pm(k)$ is the dimension of the kernel of the corresponding composition of A and B. The fact that the condition in the theorem is necessary, follows from this interpretation. $\square$

8.8. A parabolic subalgebra of $gl(V)$ is the stabilizer of a generalized flag

$$(0) = W_0 \subset W_1 \subset \cdots \subset W_k = V, \tag{8.8.1}$$

so that $W_i \neq W_{i+1}$. Fix complementary spaces V_i,

$$W_i = W_{i-1} + V_i, \qquad i > 0. \tag{8.8.2}$$

They determine a Levi component

$$\mathfrak{l} \cong gl(V_1) \times \cdots \times gl(V_k). \tag{8.8.3}$$

8.9. In order to get a θ-stable parabolic subalgebra, one needs to asume that the W_i are stable under θ, or equivalently that the restriction of the hermitian form to each W_i is nondegenerate. In this case we may assume that the V_i are θ-stable as well, and let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be the corresponding parabolic subalgebra of $gl(V)$. If we denote the signature of V_i by (p_i, q_i), then

$$\mathfrak{l}_0 \cong u(p_1, q_1) \times \cdots \times u(p_k, q_k). \tag{8.9.1}$$

8.10. To get the complexification of a *real* parabolic subalgebra, start with a partial flag

$$(0) = W_0 \subset \cdots \subset W_k \tag{8.10.1}$$

such that the hermitian form is trivial when restricted to W_k, and complete it to

$$(0) = W_0 \subset \cdots \subset W_k \subset W_k^* \subset \cdots \subset W_0^* = V \tag{8.10.2}$$

Choose transverse spaces

$$W_i = W_{i-1} + V_i, \qquad W_i^* = W_{i-1}^* + V_i^*, \qquad W_k^* = W_k + V_0. \tag{8.10.3}$$

They determine a Levi component

$$\mathfrak{l} \cong gl(V_1) \times \cdots \times gl(V_k) \times gl(V_0) \times gl(V_k^*) \times \cdots \times gl(V_1^*), \tag{8.10.4}$$

so that

$$\mathfrak{l}_0 \cong gl(V_1, \mathbb{C}) \times \cdots \times gl(V_k, \mathbb{C}) \times u(p_0, q_0). \tag{8.10.5}$$

where (p_0, q_0) is the signature of V_0.

8.11. Let now $\mathfrak{q}$ be a maximal θ stable parabolic subalgebra corresponding to the flag $W_1 = V_1 \subset W_2 = V_1 + V_2 = V$. Let $e \in gl(V_2) \subset \mathfrak{l}$ be a θ stable nilpotent element. Note that

$$\mathfrak{u} \cong \mathrm{Hom}(V_2, V_1). \tag{8.11.1}$$

Write $n_i := \dim V_i$. Let $E = e + X$, with $X \in \mathfrak{u}$. Write $\theta = \theta_1 + \theta_2$ with $\theta_i \in End(V_i)$. Then $X\theta_2 = -\theta_1 X$ and $\theta_1 e = -e\theta_2$. Decompose

$$V_2 = \bigoplus W_i^+ \ \oplus \ \bigoplus W_j^- \tag{8.11.2}$$

where $W_i^+,\ W_j^-$ are θ stable representations of a Lie triple containing e, and the eigenvalue of θ_1 on the highest weight $v_i^+,\ v_j^-$ is 1 and -1 respectively. Order the $W_i,\ W_j$ in decreasing order of their dimensions.

Proposition. *The signature $(A_+(k),\ A_-(k))$ of E^k satisfies*

$$\begin{aligned} A_+(k) \geq &\dim V_{1,+} + a_+(k-1) + \\ &+ \max(0, \#\{\ i \mid \dim W_i \geq k, \epsilon_i = (-1)^{k-1}\} - \dim V_{1,(-1)^k}), \\ A_-(k) \geq &\dim V_{1,-} + a_-(k-1) + \\ &+ \max(0, \#\{\ i \mid \dim W_i \geq k, \epsilon_i = (-1)^{k}\} - \dim V_{1,(-1)^{k-1}}). \end{aligned}$$

Proof. Since $E^k = e^k + Xe^{k-1}$, V_1 is always in its kernel. An element $v \in V_2$, is in the kernel of E^k if $e^{k-1}v$ is in the kernel of X as well as e. The intersection of the image of e^{k-1} with the kernel of e is the span of the highest vectors $v_i \in W_i$ with $\dim W_i \geq k$. The claim follows. □

8.12. We now construct an E such that the inequalities in proposition 8.11 are equalities.

For any integers $a,\ b$, let

$$\mathbb{K}_a^+ := span\{v_i^+ \ : \ i \leq a\}, \qquad \mathbb{K}_b^- := span\{v_j^+ \ : \ j \leq b\}. \tag{8.12.1}$$

Note that

$$X(\mathbb{K}_a^+) \subset V_1^-, \qquad X(\mathbb{K}_b^-) \subset V_1^+. \tag{8.12.2}$$

Theorem. *Let $E = e + X$, and with notation as in 8.12.2, choose X such that it is nonsingular on $\mathbb{K}_{a,b}^{\pm}$ for as large an a and b as possible. Then $\mathcal{O}(E) = ind_{\mathfrak{q}}^{\mathfrak{g}} e$.*

Proof. From the proposition it follows that the $a_\pm^k$ of any element in $e + (\mathfrak{u} \cap \mathfrak{s})$ are minimal when they are equal to the RHS of proposition 8.11. Theorem 8.7 implies that if a nilpotent element achieves this minimum, its orbit contains any other $e + X$ in its closure. Thus its has maximal dimension among all orbits meeting $e + (\mathfrak{u} \cap \mathfrak{s})$ and so the claim follows from the observation at the end of 8.5. □

This theorem implies the following algorithm for computing the induced orbit in the case $\mathfrak{g}_0 \cong u(p,q)$. Suppose the signature of V_1 is (a_+, a_-). Then add a_+ $+$'s to the beginning of largest possible rows starting with a $-$ and

a_- $-$'s to the largest possible rows starting with a $+$. If a_+ is larger than the number of rows starting with $-$, add a new row of size 1 starting with $+$. The similar rule applies to a_-.

If $e \in gl(V_1)$, the analogous procedure applies, but the a_+ $+$'s are added to the end of the largest possible rows finishing in $-$ and a_- $-$'s to the end of the largest possible rows finishing in $+$.

8.13. Suppose $\mathfrak{q}$ is the complexification of a real parabolic subalgebra corresponding to the flag $(0) \subset V_1 \subset V_1 + V_0 \subset V_1 + V_0 + V_1^*$, and let $e \subset gl(V_0)$.

Theorem. *The tableau of an orbit $\mathcal{O}(E_i)$ in (8.4.1) is obtained from the tableau of e by adding 2 to $\dim V_1$ of the largest rows leaving the signs unchanged.*

Proof. We use (8.4.2). Let $\alpha \in \mathrm{Hom}[V_1, V_1^*] \oplus \mathrm{Hom}[V_1^*, V_1]$ be nondegenerate such that $\alpha^2 = Id \oplus Id$, and extend it to an endomorphism $\xi \in gl(V)$ so that its restriction to V_0 is zero. Then $[\xi, e] = 0$, so $t\xi + e$ is a Jordan decomposition. Let

$$P(X) = X^m + a_{m-1}X^{m-1} + \cdots + a_0 \tag{8.13.1}$$

be any polynomial. Suppose $t_i \in \mathbb{R}$ are such that $t_i \to 0$, and assume there are $g_i \in K$ such that $t_i g_i(\xi + e)g_i^{-1} \to E$. Then

$$\ker t_i^m P(g_i(\xi + e)g_i^{-1}) \cong \ker P(\xi + e). \tag{8.13.2}$$

On the other hand,

$$\begin{aligned} t_i^m P(g_i(\xi + e)g_i^{-1}) &= [t_i g_i(\xi + e)g_i^{-1}]^m + \\ &+ a_{m-1}t_i[t_i g_i(\xi + e)g_i^{-1}]^{m-1} + \cdots + t_i^m Id \to E^m. \end{aligned} \tag{8.13.3}$$

Thus

$$\dim\ker E^m \mid_{V_\pm} \geq \dim\ker P(\xi + e) \mid_{V_\pm} . \tag{8.13.4}$$

Choosing $P(X) = (X^2 - 1)X^n$, we conclude that E must be nilpotent. Choosing $P(X) = X^m,\ (X \pm 1)X^{m-1}$ or $P(X) = (X^2 - 1)X^{m-2}$, we can bound the dimensions of $\ker E^m \mid_{V_\pm}$ to conclude that it must be in the closure of one of the nilpotent orbits in the theorem. The fact that these nilpotent orbits are indeed induced from e, follows by a direct calculation which we omit. □

8.14. **sp(V).** Suppose $\mathfrak{g} \cong sp(V_0)$, where $(V_0, \langle\ ,\ \rangle)$ is a real symplectic vector space of dimension n. The complexification $(V, \langle\ ,\ \rangle)$ admits a complex conjugation $^-$, and we define a nondegenerate hermitian form

$$(v, w) := \langle v, \overline{w}\rangle \tag{8.14.1}$$

which is of signature (n, n). Denote by $u(n, n)$ the corresponding unitary group. Since $sp(V_0)$ stabilizes $(\ ,\ )$, it embeds in $u(n, n)$, and the Cartan involutions are compatible. The results of sections 8.1-8.3 together with section 8.6 imply the following classification of nilpotent orbits of $sp(V_0)$ or equivalently θ-stable nilpotent orbits.

(1) To each orbit we assign a tableau so that every odd part occurs an even number of times. Rows of equal size are interchangeable.
(2) The entries in each row alternate $+$ or $-$. Odd sized rows occur in pairs, one starting with $+$ the other with $-$.

A parabolic sublagebra of $sp(V)$ is the stabilizer of a flag of isotropic subspaces

$$(0) = \mathcal{W}_0 \subset \cdots \subset \mathcal{W}_k, \tag{8.14.2}$$

so that the symplectic form restricts to 0 on $\mathcal{W}_k$. As before, complete this to a flag

$$(0) = \mathcal{W}_0 \subset \cdots \subset \mathcal{W}_k \subset \mathcal{W}_k^* \subset \cdots \subset \mathcal{W}_0^* = V. \tag{8.14.3}$$

We choose transverse spaces

$$\mathcal{W}_i = \mathcal{W}_{i-1} + V_i, \quad \mathcal{W}_k^* = \mathcal{W}_k + \mathcal{W}, \quad \mathcal{W}_{i-1}^* = \mathcal{W}_i^* + V_i^* \tag{8.14.4}$$

in order to fix a Levi component. We get

$$\mathfrak{l} \cong gl(V_1) \times \cdots \times gl(V_k) \times sp(\mathcal{W}). \tag{8.14.5}$$

If we assume that V_i, $\mathcal{W}$ are θ-stable, then the corresponding parabolic subalgebra is θ-stable as well and the real points of the Levi component are

$$\mathfrak{l}_0 \cong u(p_1, q_1) \times \cdots \times u(p_k, q_k) \times sp(\mathcal{W}_0). \tag{8.14.6}$$

where (p_i, q_i) is the signature of V_i. The parabolic subalgebra corresponding to 8.14.4 in $gl(V)$ satisfies

$$\mathfrak{l}' \cong u(p_1, q_1) \times \cdots \times u(p_k, q_k) \times u(n_0, n_0) \times u(q_k, p_k) \times \cdots \times u(q_1, p_1). \tag{8.14.7}$$

For a maximal θ-stable parabolic subalgebra, the Levi component $\mathfrak{l}$ satisfies $\mathfrak{l} \cong u(p_1, q_1) \times sp(\mathcal{W}_0)$. Let $e \in sp(W)$ be a θ-stable nilpotent element. The algorithm for induced nilpotent orbits in section 8.9 implies the following for $ind_{\mathfrak{l}}^{\mathfrak{g}}(e)$.

(1) add p $+$'s to the beginning of the longest possible rows starting with $-$'s, and q $-$'s to the beginning of the longest possible rows starting with $+$'s.
(2) add q $+$'s to the ending of the longest possible rows starting with $-$'s, and p $-$'s to the beginning of the longest possible rows starting with $+$'s.

Unlike in the complex case, the result is automatically a partition for a nilpotent element in $sp(V)$.

For a maximal ρ-stable parabolic subalgebra, we must assume that $\overline{V_1} = V_1$, $\overline{\mathcal{W}} = \mathcal{W}$. Let $V_{1,0}$ and $\mathcal{W}_0$ be their real points. The Levi component satisfies

$$\mathfrak{l}_0 \cong gl(V_{1,0}) \times sp(\mathcal{W}_0). \tag{8.14.8}$$

The results in section 8.13 imply the following algorithm for real induction.

(1) add 2 to $\dim V_1$ largest possible rows of e leaving the signs unchanged.

(2) Suppose $\dim V_1$ is odd and the last row that would be increased by 2 is odd size as well. In this case there is a pair of rows of this size, one starting with $+$ the other with $-$. In this case increase these two rows by one each leaving the sign unchanged.

8.15. **so(p,q).** Suppose $\mathfrak{g} \cong so(V_0)$, where $(V_0, \langle\ ,\ \rangle)$ is a real nondegenerate quadratic space of signature (p,q). The complexification admits a hermitian form $\langle\ ,\ \rangle$ with signature (p,q) as well as a complex nondegenerate quadratic form $(\ ,\ )$. The form $\langle\ ,\ \rangle$ gives an embedding of $o(p,q)$ into $u(p,q)$ with compatible Cartan involutions. The results of sections 8.1-8.3 together with section 8.6 imply the following classification of nilpotent orbits of $so(V_0)$ or equivalently θ-stable nilpotent orbits.

(1) To each orbit we assign a tableau so that every even part occurs an even number of times. Rows of equal size are interchangeable.
(2) The entries in each row alternate $+$ or $-$. Even sized rows occur in pairs, one starting with $+$ the other with $-$.
(3) When all the rows have even sizes, there are two nilpotent orbits denoted I and II.

A parabolic sublagebra of $so(V)$ is the stabilizer of a flag of isotropic subspaces

$$(0) = \mathcal{W}_0 \subset \cdots \subset \mathcal{W}_k, \tag{8.15.1}$$

so that the quadratic form restricts to 0 on $\mathcal{W}_k$. As before, complete this to a flag

$$(0) = \mathcal{W}_0 \subset \cdots \subset \mathcal{W}_k \subset \mathcal{W}_k^* \subset \cdots \subset \mathcal{W}_0^* = V. \tag{8.15.2}$$

We choose transverse spaces

$$\mathcal{W}_i = \mathcal{W}_{i-1} + V_i, \quad \mathcal{W}_k^* = \mathcal{W}_k + \mathcal{W}, \quad \mathcal{W}_{i-1}^* = \mathcal{W}_i^* + V_i^* \tag{8.15.3}$$

in order to fix a Levi component,

$$\mathfrak{l} \cong gl(V_1) \times \cdots \times gl(V_k) \times so(W). \tag{8.15.4}$$

To get a θ-stable parabolic subalgebra we must assume V_i, W are θ-stable and so $\overline{V_i} = V_i^*$, $\overline{W} = W$. If the signature of V_i with respect to $\langle\ ,\ \rangle$ is (p_i, q_i), and that of W is (p_0, q_0), then

$$\mathfrak{l}_0 \cong u(p_1, q_1) \times \cdots \times u(p_k, q_k) \times so(p_0, q_0). \tag{8.15.5}$$

The parabolic subalgebra corresponding to 8.15.2 in $gl(V)$ satisfies

$$\mathfrak{l}' \cong u(p_1,q_1) \times \cdots \times u(p_k,q_k) \times u(p_0,q_0) \times u(p_k,q_k) \times \cdots \times u(p_1,q_1). \tag{8.15.6}$$

For a maximal θ-stable parabolic subalgebra, the Levi component $\mathfrak{l}$ satisfies $\mathfrak{l} \cong u(p_1, q_1) \times so(\mathcal{W}_0)$. Let $e \in so(W)$ be a θ-stable nilpotent element. The algorithm for induced nilpotent orbits in section 8.9 implies the following for $ind_{\mathfrak{l}}^{\mathfrak{g}}(e)$.

(1) add p_1 $+$'s to the beginning of the longest possible rows starting with $-$'s, and q_1 $-$'s to the beginning of the longest possible rows starting with $+$'s.

(2) add p_1 $+$'s to the ending of the longest possible rows starting with $-$'s, and q_1 $-$'s to the beginning of the longest possible rows starting with $+$'s.

Unlike in the complex case, the result is automatically a partition for a nilpotent element in $so(V)$.

For a maximal ρ-stable parabolic subalgebra, we must assume that $\overline{V_1} = V_1$, $\overline{\mathcal{W}} = \mathcal{W}$. Let $V_{1,0}$ and $\mathcal{W}_0$ be their real points. The Levi component satisfies

$$\mathfrak{l}_0 \cong gl(V_{1,0}) \times so(W_0). \tag{8.15.7}$$

The results in section 8.13 imply the following algorithm for real induction.

(1) add 2 to $\dim V_1$ largest possible rows of e leaving the signs unchanged.
(2) Suppose $\dim V_1$ is even and the last row that would be increased by 2 is even size as well. In this case there is a pair of rows of this size, one starting with $+$ the other with $-$. Increase these two rows by one each leaving the sign unchanged.
(3) When there are only even sized rows and $\dim V_1$ is even as well, type I goes to type I and type II goes to type II.

9. Unitarity

As already mentioned, the unitarity of the unipotent representations in the p-adic case is done in [BM]. It amounts to the observation that the Iwahori-Matsumoto involution takes unipotent spherical representations to tempered ones.

The idea of the proof in the real case is described in [B2]. We give details of a simpler argument in the case $G = So(2n+1)$, only minor changes are required for the other cases. We will do an induction on rank.

9.1. We rely heavily on the properties of the WF-set, asymptotic support and associated variety, and their relations to primitive ideal cells and Harish-Chandra cells. We review some facts. Since this is not the main purpose of the article, we refer to [SV], [V2] and [BV1], [BV2], [B3] for the details.

Let π be an admissible $(\mathfrak{g}, K)$ module. According to [BV1], the distribution character Θ_π lifts to an invariant eigendistribution θ_π in a neighborhood of the idenitity in the Lie algebra. If $f \in C_c^\infty(U)$ for $U \subset \mathfrak{g}$ a small enough neighborhood of 0, let $f_t(X) := t^{-\dim \mathfrak{g}} f(t^{-1}X)$. Then

$$\theta_\pi(f_t) = t^{-d}[\sum c_j \widehat{\mu_{\mathcal{O}_j}}(f) + \sum_{i>0} t^i D_i(f)]. \tag{9.1.1}$$

The D_i are homogeneous invariant distributions (each D_i is tempered and the support of its Fourier transform is contained in the nilpotent cone). The $\mu_{\mathcal{O}_j}$ are invariant measures supported on real forms $\mathcal{O}_j$ of a single complex orbit $\mathcal{O}_c$, and $\mu_{\mathcal{O}_j}$ is the Liouville measure on the nilpotent orbit associated to the symplectic form induced by the Cartan-Killing form. Furthermore $d = \dim_{\mathbb{C}} \mathcal{O}_c/2$, and the number c_j is called the multiplicity of $\mathcal{O}_j$ in the leading term of the expansion. The closure of the union of the supports of the Fourier transforms of all the terms occuring in 9.1.1 is called the wave front set, denoted $WF(\pi)$.

Alternatively, [V2] attaches to each π a combination of θ-stable orbits with integer coefficients

$$AV(\pi) = \sum a_j \mathcal{O}_j, \tag{9.1.2}$$

where $\mathcal{O}_j$ are K-orbits in $\mathfrak{s}$. The main [SV] is that the orbits and multiplicities in 9.1.1 and 9.1.2 correspond via theorem 8.3, precisely formula 8.3.2, and the multiplicities are the same *i.e.* $c_j = a_j$. The main point of algorithms in section 8 is that they compute the associated variety of an induced representation as a set, which we denote by $WF(\pi)$. These multiplicities are computed in the real setting in [B4] theorem 5.0.7; the formula is as follows. Let $v_j \in \mathcal{O}_j$ and $v_{ij} = v_j + X_{ij}$ be representatives of the induced orbits from $\mathcal{O}_j$. If $AV(\pi) = \sum c_j \mathcal{O}_j$, then $AV(ind_{\mathfrak{p}}^{\mathfrak{g}}(\pi))$ is

$$\sum_{i,j} \frac{|C_G(v_{ij})|}{|C_P(v_{ij})|} \mathcal{O}_{ij} \tag{9.1.3}$$

This is the only place where we use [SV]. The multiplicities are straightforward to compute for real induction in terms of real orbits, it is the passage to $AV(\pi)$ that is nontrivial.

9.2. Fix a regular integral infinitesimal character χ_{reg}. Denote by $\mathcal{G}(\chi_{reg})$ the Grothendieck group of the category of $(\mathfrak{g}, K)$ modules with infinitesimal character χ_{reg}. Recall from [V2] (and references therein) that there is an action of the Weyl group on $\mathcal{G}(\chi_{reg})$, called the *coherent continuation action.* Then $\mathcal{G}(\chi_{reg})$ decomposes into a direct sum according to blocks $\mathcal{B}$,

$$\mathcal{G}(\chi_{reg}) = \bigoplus \mathcal{G}_{\mathcal{B}}(\chi_{reg}). \tag{9.2.1}$$

We give the explicit description of the representation.
Type B: The Cartan subgroups are parametrized by four integers $(p, q, 2s, r)$, satisfying $p + q + 2s + r = n$. The corresponding representation is

$$\sum_{\sigma \in \widehat{W}_{2s}} Ind_{W_p \times W_q \times W_{2s} \times S_t}^{W_n} [sgn \otimes sgn \otimes \sigma \otimes triv]. \tag{9.2.2}$$

The sum is over the $\sigma = \tau \times \tau$ where τ is a partition of s. The representation σ is labelled by dots, sign by r or r', and *triv* by c. Recall also the well known formula

$$Ind_{S_n}^{W_n}(triv) = \sum_{a+b=n} (a) \times (b) \tag{9.2.3}$$

To induce we add r and r' at most one to each row to τ_R, and c at most one to each column to both τ_L and τ_R the total number being t.

Type C: The Cartan subgroups are parametrized by four integers $(t, 2s, p, q)$, satisfying $p + q + 2s + t = n$. The corresponding representation is

$$\sum_{\sigma\in\widehat{W}_{2s}} Ind^{W_n}_{S_t\times W_{2s}\times W_p\times W_q}[sgn\otimes\sigma\otimes triv\otimes triv]. \tag{9.2.4}$$

The sum is over the $\sigma = \tau\times\tau$ where τ is a partition of s. The notation is set up to take the duality in [V2] of types B and C into account. So we write r for the sign representation of S_t, and c and c' for the trivial representation of $W_p,\ W_q$. We denote the rows of τ_L as $0, 2, \dots, 2m$ and the rows of τ_R as $1, 3, \dots, 2m-1$ to conform to the notation of the special symbol

$$\begin{pmatrix} r_0 & & r_2+1 & & & \dots & & & r_{2m}+m \\ & r_1 & & r_3+1 & \dots & & r_{2m-1}+m-1 & & \end{pmatrix} \tag{9.2.5}$$

Type D: The Cartan subgroups are parametrized by integers $(t, u, 2s, p, q)$, $p + q + 2s + t + u = n$. There are actually two Cartan subgroups for each $s > 0$. The corresponding representation is

$$\sum_{\sigma\in\widehat{W}'_{2s}} Ind^{W'_n}_{W_p\times W_q\times W'_{2s}\times W_t\times W_u}[sgn\otimes sgn\otimes\sigma\otimes triv\otimes triv]. \tag{9.2.6}$$

The sum is over the $\sigma = \tau\times\tau$ where τ is a partition of s. We label the σ by dots, trivial representations by c and c' and the sgn representations by r and r'. These are added to the left τ_L when inducing.

In this case we denote the rows of τ_L as $0, 2, \dots, 2m-2$ and the rows of τ_R as $1, 3, \dots 2m-1$. This conforms to the special symbol notation

$$\begin{pmatrix} r_0 & r_2+1 & \dots & r_{2m-2}+m-1 \\ r_1 & r_3+1 & \dots & r_{2m-1}+m-1 \end{pmatrix} \tag{9.2.7}$$

Let $\mathfrak{h}_a \subset \mathfrak{g}$ be an abstract Cartan subalgebra and let Π_a be a set of (abstract) simple roots. For each irreducible representation $\mathcal{L}(\gamma)$, denote by $\tau(\gamma)$ the tau-invariant as defined in [V2]. Given a block $\mathcal{B}$ and disjoint orthogonal sets $S_1,\ S_2 \subset \Pi_a$, define

$$\mathcal{B}(S_1, S_2) = \{\gamma\in\mathcal{B} | S_1 \subset \tau(\gamma),\ S_2\cap\tau(\gamma) = \emptyset\}\ . \tag{9.2.8}$$

If in addition we are given a nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$, we can also define

$$\mathcal{B}(S_1, S_2, \mathcal{O}) = \{\gamma\in\mathcal{B}(S_1, S_2) |\ WF(\mathcal{L}(\gamma)) \subset \overline{\mathcal{O}}\}\ . \tag{9.2.9}$$

Consider the case of a complex algebra $\mathfrak{g}$ viewed as a real Lie algebra. Then the case $S_1,\ S_2 = \emptyset$ is called the double cone $\mathcal{C}(\mathcal{O})$. The double cell corresponding to $\mathcal{O}$ will be denoted $\overline{\mathcal{C}}(\mathcal{O})$.

Let $W_i = W(S_i)$, and define

$$\begin{aligned} m_S(\sigma) &= [\sigma : Ind^W_{W_1\times W_2}(Sgn\otimes Triv)],\\ m_{\mathcal{B}}(\sigma) &= [\sigma : \mathcal{G}_{\mathcal{B}}(\chi_{reg})]\ . \end{aligned} \tag{9.2.10}$$

Theorem (Vogan).

$$|\mathcal{B}(S_1, S_2, \mathcal{O})| = \sum_{\sigma\otimes\sigma\in\mathcal{C}(\mathcal{O})} m_{\mathcal{B}}(\sigma) m_S(\sigma) \ .$$

Recall $\lambda = \lambda_{\check{\mathcal{O}}}$. Then λ defines a set S_2 by

$$S_2 = S(\lambda) = \{\alpha \in \Pi_a | (\alpha, \lambda) = 0\} \ . \tag{9.2.11}$$

Then the special unipotent representations attached to $\check{\mathcal{O}}$ are defined to be

$$Unip(\check{\mathcal{O}}) = \bigcup_{\mathcal{B}} \mathcal{B}(\emptyset, S(\lambda), \mathcal{O}) \tag{9.2.12}$$

In the classical groups case, $m_{\mathcal{B}}(\sigma)$ is straightforward to compute. For the special unipotent case, $m_S(\sigma)$ equals 0 except for the representations occuring in the corresponding left cell $\overline{\mathcal{C}}^L(\mathcal{O})$ when it is 1. The representations are in 1-1 correspondence with the conjugacy classes in Lusztig's quotient of the component group $A(\overline{\check{\mathcal{O}}})$. See [BV2] for details.

Theorem (2).

$$|Unip(\check{\mathcal{O}})| = \sum_{\mathcal{B}} \sum_{\sigma\otimes\sigma\in\overline{\mathcal{C}}^L(\mathcal{O})} m_{\mathcal{B}}(\sigma) \ .$$

Definition. *We say that a nilpotent orbit $\mathcal{O}$ is* smoothly cuspidal *if it satisfies*

Type B, D: *all odd sizes occur an even number of times,*
Type C: *all even sizes occur an even number of times.*

For $\mathcal{O}(\mathbb{R})$, a real form of $\mathcal{O}$, write $A(\mathcal{O}(\mathbb{R}))$ for its (real) component group.

Proposition. *For smoothly cuspidal orbits, $A(\check{\mathcal{O}}) = \overline{A(\check{\mathcal{O}})}$. In particular $|\overline{\mathcal{C}}^L(\mathcal{O})| = |A(\check{\mathcal{O}})|$. Furthermore,*

$$|Unip(\check{\mathcal{O}}))| = |A(\check{\mathcal{O}})| \sum_{\mathcal{O}(\mathbb{R})} |A(\mathcal{O}(\mathbb{R}))|$$

where the sum is over all real forms. The set $Unip_{\mathcal{B}}(\mathcal{O})$, consisting of the unipotent representations in the block containing the spherical representation, satisfy

$$|Unip_{\mathcal{B}}(\mathcal{O})| = |\text{ real forms of } \mathcal{O}| \cdot |A(\check{\mathcal{O}})|.$$

Proof. The first part is theorem 5.3 in [B2]. It consists of a calculation of multiplicities in the coherent continuation representation. The same calculation yields the second statement. We omit further details which can be found in [B5]. □

9.3. Two representations $\pi,\ \pi'$ are said to be in the same Harish-Chandra cell if there are finite dimensional representations $F,\ F'$ such that π' is a factor of $\pi\otimes F$ and π a factor of $\pi'\otimes F'$. In this case $WF(\pi)=WF(\pi')$. We say that a Harish-Chandra cell is attached to a complex orbit $\mathcal{O}$ if $\overline{\operatorname{Ad} G(WF(\pi))}=\overline{\mathcal{O}}$. The set of representations in a Harish-Chandra cell give rise to a representation of the (complex) Weyl group.

Theorem ([McG]). *In the classical groups $Sp(n)$, $So(p,q)$, each Harish-Chandra cell is of the form $\overline{\mathcal{C}}^L(\mathcal{O})$.*

9.4. Consider the spherical irreducible representation $L(\check{\mathcal{O}})$ corresponding to a nilpotent orbit $\check{\mathcal{O}}$ in $sp(n)$. If the orbit $\check{\mathcal{O}}$ meets a proper Levi component $\check{\mathfrak{m}}$, then $L(\check{\mathcal{O}})$ is a subquotient of a representation which is unitarily induced from a unipotent representation on $\mathfrak{m}$. By induction, $L(\check{\mathcal{O}})$ is unitary. Thus we assume that $\check{\mathcal{O}}$ does not meet any proper Levi component which means

$$\check{\mathcal{O}}=(2x_0,\dots,2x_{2m}),\qquad 0\le x_0<\dots<x_i<x_{i+1}<\dots<x_{2m}. \tag{9.4.1}$$

Because of assumption (9.4.1), the WF-set of $L(\check{\mathcal{O}})$ satisfies the property that

$$\overline{\operatorname{Ad} G(WF(L(\check{\mathcal{O}}))}$$

is the closure of the special orbit (in the sense of Lusztig) dual to $\check{\mathcal{O}}$. This is the orbit $\mathcal{O}$ with partition

$$(\underbrace{1,\dots,1}_{r_1},\underbrace{2,\dots,2}_{r_2},\dots,\underbrace{2m,\dots,2m}_{r_{2m}},\underbrace{2m+1,\dots,2m+1}_{r_{2m+1}}), \tag{9.4.2}$$

where $r_i=x_{2m-i+1}-x_{2m-i}$ and $r_{2m+1}=2x_0+1$. Every size but the largest one appears an even number of times in the partition of the nilpotent orbit $\mathcal{O}_c$.

Definition. *Given an orbit $\mathcal{O}$ with partition 9.4.2, we call the* split real form *the one where, for a given row size,*

> **Type C,D:** *the number of rows starting with $+$ as with $-$ is equal,*
> **Type B:** *in addition there is one more row of size $2m+1$ starting with $+$ than with $-$.*

Theorem. *The WF-set of the representation $L(\check{\mathcal{O}})$ with $\check{\mathcal{O}}$ satisfying 9.4.1 is the closure of the split real form $\mathcal{O}_{spl}$ of the (complex) orbit $\mathcal{O}$ given by 9.4.2.*

Proof. The main idea is outlined in [B2]. We use the fact that if π is a factor of π', then $WF(\pi)\subset WF(\pi')$. We do an induction on m. The claim amounts to showing that if E occurs in $WF(L(\check{\mathcal{O}}))$, then the signatures of $E,\ E^2,\dots$ are greater than the pairs

$$\begin{aligned}&(x_{2m}+1,x_{2m}),\\&(x_{2m}+x_{2m-1},x_{2m}+x_{2m-1}),\dots,(x_{2m}+\dots+x_1,x_{2m}+\dots+x_1),\\&(x_{2m}+\dots+x_1+x_0+1,x_{2m}+\dots+x_1+x_0).\end{aligned} \tag{9.4.3}$$

The statement is clear when $m = 0$; $L(\check{\mathcal{O}})$ is the trivial representation. Let $\check{\mathcal{O}}_1$ be the nilpotent orbit corresponding to

$$(2x_0, \dots, 2x_{2m-2}). \tag{9.4.4}$$

By induction, $WF(L(\check{\mathcal{O}}_1))$ is the split real form of the nilpotent orbit corresponding to the partition

$$(\underbrace{1, \dots, 1}_{r'_1}, \underbrace{2, \dots, 2}_{r'_2}, \dots, \underbrace{2m-2, \dots, 2m-2}_{r'_{2m-2}}, \underbrace{2m-1, \dots, 2m-1}_{r'_{2m-1}}), \tag{9.4.5}$$

where $r'_i = x_{2m-2-i+1} - x_{2m-2-i}$ and $r'_{2m-1} = 2x_0 + 1$. Let $\mathfrak{p}$ be the real parabolic subalgebra with Levi component $\mathfrak{g}(n - x_{2m} - x_{2m-1}) \times gl(x_{2m} - x_{2m-1})$. There is a character χ of $gl(x_{2m} - x_{2m-1})$ such that $\pi := L(\check{\mathcal{O}})$ is a factor of $\pi' := ind_{\mathfrak{p}}^{\mathfrak{g}}[L(\check{\mathcal{O}}_1) \otimes \chi]$. But by section 8, $WF(\pi')$ is in the closure of nilpotent orbits corresponding to partitions

$$(\underbrace{2, \dots, 2}_{(r_1+r_2)/2}, \dots, \underbrace{2m, \dots, 2m}_{r_{2m}}, \underbrace{2m+1, \dots, 2m+1}_{r_{2m+1}}), \; r_1 + r_2 \text{ even}, \tag{9.4.6}$$

$$(1, 1, \underbrace{2, \dots, 2}_{(r_1+r_2-1)/2}, \dots, \underbrace{2m, \dots, 2m}_{r_{2m}}, \underbrace{2m+1, \dots, 2m+1}_{r_{2m+1}}), \; r_1 + r_2 \text{ odd}. \tag{9.4.7}$$

In any case, it follows that the signatures for E^k in $WF(L(\check{\mathcal{O}}))$ are greater than the pairs

$$(a_+, a_-), \; (x_{2m} + x_{2m-1}, x_{2m} + x_{2m-1}), \dots \; . \tag{9.4.8}$$

Also, each row size greater than two and less than $2m + 1$ has an equal number that start with $+$ and $-$, and for size $2m+1$ there is one more row starting with $+$ than $-$.

The same argument with $\check{\mathcal{O}}_2$ corresponding to

$$(2x_0, \dots \widehat{2x_{2m-2}}, \widehat{2x_{2m-1}}, 2x_{2m})$$

shows that $WF(L(\check{\mathcal{O}}))$ is also contained in the closure of the nilpotent orbits with signatures

$$\begin{aligned}&(x_{2m} + 1, x_{2m}), \; (x_{2m} + 1 + a_+, x_{2m} + a_-), \\ &(x_{2m} + 1 + x_{2m_1} + x_{2m-2}, x_{2m} + 1 + x_{2m_1} + x_{2m-2}), \; \dots \; .\end{aligned} \tag{9.4.9}$$

The signs on the rows greater than 2 are as claimed. $\square$

9.5. Consider the special case when

$$x_0 = x_1 - 1 \le x_2 = x_3 - 1 \le \dots \le x_{2m-2} = x_{2m-1} - 1 \le x_{2m}. \tag{9.5.1}$$

Theorem 9.4 computes the WF-set of the spherical representation $L(\check{\mathcal{O}})$ and the results in section 8 show that

$$WF(L(\check{\mathcal{O}})) \subset ind_{\mathfrak{p}}^{\mathfrak{g}}[WF(L(\check{\mathcal{O}}_k) \times triv] \tag{9.5.2}$$

where $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ and

$$\begin{aligned} \check{\mathcal{O}}_k &= (x_0, \dots, \widehat{x_{2k} x_{2k+1}}, \dots, x_{2m}), \\ \mathfrak{m} &= gl(x_{2k} + x_{2k+1}) \times \mathfrak{g}(n - x_{2k} - x_{2k+1}). \end{aligned} \tag{9.5.3}$$

The component group $A(\mathcal{O})$ has size 2^m.

We produce 2^m irreducible representations so that WF equals the closure of $\mathcal{O}_{spl}$. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be the θ-stable parabolic such that

$$\mathfrak{l} = u(x_{2i_1+1}, x_{2i_1}) \times u(x_{2i_2}, x_{2i_2+1}) \times \cdots \times \mathfrak{g}(x_{2m}). \tag{9.5.4}$$

The derived functor modules $\mathcal{R}^i_{\mathfrak{q}}(\xi)$ from characters on $\mathfrak{l}$ have WF-set equal to $\mathcal{O}_{spl}$. To get infinitesimal character $\lambda(\check{\mathcal{O}})$, these characters can only be

$$\pm(1/2, \dots, 1/2), \tag{9.5.5}$$

on the unitary factors and trivial on $g(x_{2m})$. For each pair $(x_{2i} = a_i - 1, x_{2i+1} = a_i)$ we construct two Langlands parameters,

$$\begin{aligned} &(\underline{a_i}, \underline{a_i - 1}, \underline{a_i - 1}, \dots, \underline{1/2}, \underline{1/2}), \\ &(1/2_{nc}, \underline{a_i, -a_i + 1}, \dots, \underline{3/2, -1/2}). \end{aligned} \tag{9.5.6}$$

The notation is as follows. The vector represents a functional in a Cartan subalgebra $\mathfrak{h}$, with the standard positive roots. The various subscripts and underlinings describe the nature of the roots, compact, noncompact imaginare, real or complex. A coordinate a_c, a_{nc} which is not underlined at all, means that the corresponding short root is either compact or noncompact and the value is a. A coordinate $\underline{a}$ which is underlined denotes that the corresponding short root is real, and the value is a. A superscript $\underline{a}^{\pm}$ distinguishes whether the character on M restricted to the m_α is trivial $(+)$ or sign $(+-)$. A pair $\underline{a, b}$ which is underlined denotes that the corresponding $\epsilon_i - \epsilon_j$ is imaginary, $\epsilon_i + \epsilon_j$ is real.

Proposition. *The 2^m representations obtained by concatenating all possible parameters as in 9.5.6 with $(\underline{x_{2m} - 1/2}, \dots, \underline{1/2})$ have WF-set equal to $\mathcal{O}_{spl}$.*

Proof. We do an induction on m. The claim is clear for $m = 0$. Consider the induced module from a representation on $\mathfrak{m}$ in 9.5.3 which is a character on $gl(2x_{2k} + 1)$ and one of the modules with parameters as in 9.5.6. By induction they have WF-set equal to $\mathcal{O}_{split}$ and there are $2^m - 1$ distinct such representations. Remains to show that the parameter where all entries are as in the second part of 9.5.6 also has this property. We claim that this module is $\mathcal{R}_{\mathfrak{q}}(\chi)$ for the character where we use $+$ for all the x_{2i_j} in 9.5.5. It is enough to consider the case $i_j = j$. Write ξ for the character on $\mathfrak{l}$. The vanishing results in [KnV] also hold, so $\mathcal{R}^i_{\mathfrak{q}} = 0$ except for $i = \dim \mathfrak{u} \cap \mathfrak{k}$. The module is nonzero because

$$\mu := \xi + 2\rho(\mathfrak{u} \cap \mathfrak{s}) - \rho(\mathfrak{u}) \tag{9.5.7}$$

is dominant for $\mathfrak{k}$; the Blattner type formula implies that this K-type occurs in $\mathcal{R}^{\dim \mathfrak{u} \cap \mathfrak{k}}_{\mathfrak{q}}$. Remains to show that it has the Langlands parameter that we claimed. Assume the Lie algebra is $so(2p+1,2p)$ the other case is similar. Let $\mathfrak{h}$ be the compact Cartan subalgebra. We write the coordinates

$$(a_1,\dots,a_p \mid b_1,\dots,b_p) \tag{9.5.8}$$

where the first p coordinates before the $|$ are in the Caratn subalgebra of $so(2p+1)$ the last p coordinates are in $so(2p)$. The roots $\epsilon_i \pm \epsilon_j, \epsilon_i$ with $i,j \le p$ are all compact and so are $\epsilon_{p+k} \pm \epsilon_{p+l}$ with $k,l \le p$. The roots $\epsilon_i \pm \epsilon_{p+k},\ \epsilon_{p+k}$ are noncompact. The Langlands parameter is on the Cartan subalgebra $\mathfrak{h}_\mathfrak{q}$ where the roots $\epsilon_i - \epsilon_{p+i}$ for $i \le x_0 + \dots + x_{2m-1}$ are real. It can be as written (λ^G,ν) with

$$\lambda^G = (1/2,\dots,1/2,\underbrace{0,\dots,0}_{x_{2m}/2} \mid 1/2,\dots,1/2,\underbrace{0,\dots,0}_{x_{2m}/2}). \tag{9.5.9}$$

Then ν equals

$$a_{2m-2}(\epsilon_2-\epsilon_{p+1}) + (a_{2m-2}-1)(\epsilon_3-\epsilon_{p+2}) + \dots. \tag{9.5.10}$$

Let $\mathfrak{b}$ be the Borel subalgebra containing the Cartan subalgebra $\mathfrak{h}_\mathfrak{q}$ so that the roots $\Delta(\mathfrak{b},\mathfrak{h}_\mathfrak{q})$ are

$$\{\alpha\ :\ (\lambda^G,\alpha)>0,\ \text{or if } (\alpha,\lambda^G)=0 \text{ then } (\alpha,\nu)<0\}. \tag{9.5.11}$$

Let $\mathfrak{b}_\mathfrak{q} := \mathfrak{b}\cap\mathfrak{l}$. The parameter (λ^G,ν) determines a standard module $X_\mathfrak{l}(\lambda^G,\nu)$ for the pair $(\mathfrak{l},L\cap K)$. This standard module is a principal series which has a unique irreducible submodule which is a character on all the unitary factor and trivial on $\mathfrak{g}(x_{2m})$ (ξ from formula 9.5.7). It satisfies

$$\mathcal{R}^i_\mathfrak{q}(X_\mathfrak{l}(\lambda^G,\nu)) = \begin{cases} X(\lambda^G,\nu) & \text{if } i=\dim\mathfrak{k}\cap\mathfrak{u},\\ 0 & \text{otherwise.}\end{cases} \tag{9.5.12}$$

Thus there is a map

$$\mathcal{R}^{\dim\mathfrak{k}\cap\mathfrak{u}}_\mathfrak{q}(\xi) \longrightarrow \mathcal{R}^{\dim\mathfrak{k}\cap\mathfrak{u}}_\mathfrak{q}(X_\mathfrak{l}(\lambda^G,\nu)). \tag{9.5.13}$$

This map is nonzero because 9.5.7 is dominant so this is a bottom layer K-type. □

9.6.

Theorem. *The spherical unipotent representations $L(\check{\mathcal{O}})$ are unitary.*

Proof. There is a parabolic subalgebra $\mathfrak{p}^+$ with Levi component $\mathfrak{m}^+ := \mathfrak{g}(n)\times gl(n_1)\times\dots\times gl(n_k)$ in $\mathfrak{g}^+$ of rank $n+n_1+\dots+n_k$, such that the split form $\mathcal{O}^+_{spl}$ of

$$\mathcal{O}^+ = (1,1,3,3,\dots,2m-1,2m-1,2m+1)$$

is induced from $\mathcal{O}$ on $\mathfrak{g}(n)$, trivial on the gl's. By the results in sections 9.2-9.5, specifically proposition 9.2, there are $3^m\cdot 2^m$ unipotent representations, 3^m for the real forms of $\mathcal{O}$ and 2^m for the primitive ideal cell. We show that in this block, for each real form $\mathcal{O}_j$ there is exactly one Harish-Chandra cell characterized by the fact that $WF(\pi) = \overline{\mathcal{O}_j}$. Because of theorem 9.3 it is enough to produce one representation with this property for each orbit.

From section 9.1, each such form $\mathcal{O}_j$ is θ-stable induced from the trivial nilpotent orbit on a parabolic subalgebra with Levi component a real form of $gl(1)\times gl(3)\times\cdots\times gl(2m-1)\times\mathfrak{g}(m)$. Using the results in [KnV], for each such parabolic subalgebra, we can find a derived functor induced module from an appropriate 1-dimensional character, that is nonzero and has associated variety equal to the closure of the given real form. In fact we can construct this derived functor module at regular infinitesimal character where the fact that it is nonzero irreducibile is considerably easier. The facts listed in section 9.3 imply that there are representations in this cell which are nonzero when we apply translation functors to infinitesimal character $\lambda_{\check{\mathcal{O}}}$.

So in this block, there is a cell for each real form of $\mathcal{O}^+$, and each cell has 2^m irreducible representations with infinitesimal character $\chi_{\check{\mathcal{O}}}$. In particular for the split version, the Levi component is $u(1,0) \times u(1,2) \times u(3,2) \times \cdots \times so(m, m+1)$. For this case, section 9.5 produced exactly 2^m parameters; their lowest K-types are of the form μ_e. These are the only possible constituents of the induced from $L(\check{\mathcal{O}})$. Since the constituents of the restriction of a μ_e to a Levi component are again μ_e's, the only way $L(\check{\mathcal{O}})$ can fail to be unitary is if the form is negative on one of the K-types μ_e. But sections 6.2 and 5 show that the form is positive on the K-types μ_e of $L(\check{\mathcal{O}})$. □

10. Irreducibility

10.1. To complete the classification of the unitary dual we also need to show that the unipotent representations corresponding to the case when there is i such that $x_{i-1} = x_i = x_{i+1}$ are unitarily induced irreducible from the corresponding unipotent representation on a Levi component $G(n-2x_i) \times GL(x_i)$. This is clear in the p-adic case from the work of Kazhdan-Lusztig ([BM]), but somewhat involved in the real case. It follows from the following theorem. We will give a different proof in the next sections.

Theorem ([B5]). *The associated variety of a spherical representation $L(\check{\mathcal{O}})$ is given by the sum with multiplicity one of the following nilpotent orbits.*

- **Type B, D:** *On the odd sized rows, the difference between the number of $+$'s and number of $-$'s is 1, 0 or -1.*
- **Type C:** *On the even sized rows, the difference between the number of $+$'s and number of $-$'s is 1, 0 or -1.*

10.2. We need to study the induced modules from the trivial module on $\mathfrak{m} \subset \mathfrak{g}(n)$ where $\mathfrak{m} \cong gl(n)$, or $\mathfrak{m} \cong gl(a) \times \mathfrak{g}(b)$.

Type B. The nilpotent orbit $\check{\mathcal{O}}$ corresponds to the partition $2x_0 = 2x_1 = 2a$, in $sp(n,\mathbb{C})$. The infinitesimal character is $(-a+1/2, \dots, a-1/2)$ and the nilpotent orbit $\mathcal{O}$ corresponds to $(1, 1, \underbrace{2, \dots 2}_{2a-2}, 3)$. There are three real forms of this nilpotent orbit corresponding to

$$\begin{array}{ccc}
+\ -\ + & +\ -\ + & -\ +\ - \\
+\ - & +\ - & +\ - \\
-\ + & -\ + & -\ + \\
\vdots\ \vdots & \vdots\ \vdots & \vdots\ \vdots \\
+\ - & +\ - & +\ - \\
-\ + & -\ + & -\ + \\
+ & + & + \\
+ & - & +
\end{array} \tag{10.2.1}$$

Only the last two correspond to representations in the split group. There are eight total unipotent representations. Their parameters are as follows.

$$\begin{array}{ll}
(\underline{(a-1/2)^+}, \underline{(a-1/2)^-}, \dots, \underline{3/2^+}, \underline{3/2^-}, & \underline{1/2^{\pm}}, 1/2_{nc}) \\
((\underline{a-1/2})^+, (\underline{a-1/2})^+, \dots, (\underline{3/2})^+, (\underline{3/2})^+, & (\underline{1/2})^+, (\underline{1/2})^+) \\
((\underline{a-1/2})^-, \underline{a-1/2}^-, \dots, (\underline{3/2})^-, (\underline{3/2})^-, & (\underline{1/2})^-, (\underline{1/2})^-) \\
((\underline{a-1/2})^+, (\underline{a-1/2})^+, \dots, (\underline{3/2})^+, (\underline{3/2})^+, & (\underline{1/2})^+, 1/2_{nc}) \\
((\underline{a-1/2})^-, \underline{a-1/2}^-, \dots, (\underline{3/2})^-, (\underline{3/2})^-, & (\underline{1/2})^-, 1/2_{nc}) \\
(\underline{a-1/2})^+, (\underline{a-1/2})^-, \dots, (\underline{3/2})^+, (\underline{3/2})^-, & (\underline{1/2})^{\pm}, 1/2_c)
\end{array} \tag{10.2.2}$$

The superscripts above the underlined coordinates refer to the character of the corresponding M_α. The first two parameters are on $so(2a+2, 2a-1)$, the others on $So(2a+1, 2a)$. The lowest K-types of the parameters are

$$\begin{array}{c}
(\underbrace{0, \dots, 0}_{a+1} \mid \underbrace{1, \dots, 1}_{a-2}) \\
(\underbrace{0, \dots, 0}_{a} \mid \underbrace{0, \dots, 0}_{a}) \\
(\underbrace{0, \dots, 0}_{a} \mid 2, \underbrace{0, \dots, 0}_{a-1}) \\
(\underbrace{0, \dots, 0}_{a} \mid 2, \underbrace{1, \dots, 1}_{a-1})
\end{array} \tag{10.2.3}$$

Each line represents two K-types which are the same on the Lie algebra, but differ on the center of the group.

We will study the induced from the trivial module on $\mathfrak{m} \cong gl(2a)$. The WF-set is the middle nilpotent orbit in (10.2.1). So only two of the middle parameters are relevant.

Type C. The nilpotent orbit $\check{\mathcal{O}}$ corresponds to the partition $2x_0 = 2x_1 = 2a+1 < 2x_2 = 2b+1$ in $so(n, \mathbb{C})$. The infinitesimal character is

$$(-a, \dots, a)(-b, \dots, -1) \tag{10.2.4}$$

The nilpotent orbit $\mathcal{O}$ is induced from the trivial one on $gl(2a+1)\times\mathfrak{g}(b)$ and corresponds to

$$(\underbrace{1,\dots,1}_{2b-2a-2},2,2,\underbrace{3,\dots,3}_{2a}). \tag{10.2.5}$$

There are three real forms,

$$\begin{array}{ccccccccccc}
+ & - & + & & + & - & + & & + & - & + \\
- & + & - & & - & + & - & & - & + & - \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
+ & - & + & & + & - & + & & + & - & + \\
- & + & - & & - & + & - & & - & + & - \\
+ & - & & & + & - & & & - & + & \\
+ & - & & & - & + & & & - & + & \\
+ & & & & + & & & & + & & \\
- & & & & - & & & & - & & \\
\vdots & & & & \vdots & & & & \vdots & & \\
+ & & & & + & & & & + & & \\
- & & & & - & & & & - & &
\end{array} \tag{10.2.6}$$

Again there are 8 unipotent representations and 6 in the block of the spherical one. Their parameters are as follows.

We will study the module which is real induced from the trivial representation on $\mathfrak{m}\cong gl(2a+1)\times\mathfrak{g}(b)$. The WF-set is the middle one in 10.2.6

Type D. The nilpotent orbit $\check{\mathcal{O}}$ corresponds to the partition $2x_0 = 2x_1 = 2a+1$ in $so(n,\mathbb{C})$. The infinitesimal character is $(-a,\dots,a)$. The real forms of the nilpotent orbit $\mathcal{O}$ are

$$\begin{array}{cc}
+ & - \\
- & + \\
\vdots & \vdots \\
+ & - \\
- & +
\end{array} \tag{10.2.7}$$

There are two nilpotent orbits with this partition labelled I, II. Each of them is induced from $\mathfrak{m}\cong gl(2a)$. We will study these induced modules.

Proposition. *The composition factors of the induced module from the trivial representation on $\mathfrak{m}$ all have spherically relevant lowest K-types. In particular, the induced module is generated by spherically relevant K-types. Precisely,*

Type B: *the representation is generated by the μ_e,*

Type C: *the representation is generated by the μ_o,*

Type D: *the representation is generated by $\mu_e(0)=\mu_o(0)$.*

Proof. This follows from the description of the parameters of the unipotent representations and their WF-sets given above. $\square$

Corollary. *In type B, the induced module* $I(-a+1/2,\dots,a-1/2,-a+1/2,\dots a-1/2)$ *has exactly two composition factors with lowest K-types* μ_e.

Proof. The WF-set of the induced module is a single orbit, and the multiplicity is 2. ☐

10.3. We now prove the irreducibility result mentioned at the beginning of the section in the case of type B; the other cases are similar. Let $\check{\mathcal{O}}_1$ be the nilpotent orbit where we have removed one string of size $x_{i-1} := a$. Let $\mathfrak{m} := gl(a)\times\mathfrak{g}(n-a)$. Then $L(\check{\mathcal{O}})$ is the spherical subquotient of the induced representation

$$I(a, L(\check{\mathcal{O}}_1)) := Ind_{\mathfrak{m}}^{\mathfrak{g}}[(-a+1/2,\dots,a-1/2)\otimes L(\check{\mathcal{O}}_1)]. \tag{10.3.1}$$

It is enough to show that if a paramter is unipotent and satisfies $x_{i-1}=x_i=x_{i+1}=a$, then $I(a,L(\check{\mathcal{O}}_1))$ is generated by its K-types of the form μ_e. This is because of theorem 5.3, the spherical irreducible subquotient is generated by the same set.

First, we reduce to the case when there are no $0<x_j<a$. Let ν be the dominant parameter of $L(\check{\mathcal{O}})$, and assume i is the smallest index so that $x_{i-1}=a$. There is an intertwining operator

$$X(\nu)\longrightarrow I(1/2,\dots,x_0-1/2;\dots;1/2,\dots,x_{i-2}-1/2;\nu') \tag{10.3.2}$$

where I is induced from $gl(x_0)\times\cdots\times gl(x_{i-2})\times g(n-\sum_{j<i-1}x_j)$ with characters on the gl's corresponding to the strings in (10.3.2) and the irreducible module $L(\nu')$ on $\mathfrak{g}(n-\sum_{j<i-1}x_j)$. The intertwining operator is onto, and thus the induced module is generated by its spherical vector. By the induction hypothesis, the induced module from $(-a+1/2,\dots,a-/2)\otimes L(\nu'')$ on $gl(2a)\times\mathfrak{g}(n-\sum_{j\le i}x_j)$ is irreducible. But

$$\begin{aligned}&I(1/2,\dots,x_0-1/2;\dots;1/2,\dots,x_{i-2}-1/2;-a+1/2,\dots,a-1/2;\nu'')\cong\\&I(-a+1/2,\dots,a-1/2;1/2,\dots,x_0-1/2;x\dots;1/2,\dots,x_{i-2}-1/2,\nu'')\end{aligned} \tag{10.3.3}$$

This module maps by an intertwining operator onto $I(a,L(\check{\mathcal{O}}_1))$, as required.

So we have reduced to the case when

$$\begin{aligned}x_0=x_1=x_2=a,&\qquad\text{or}\\x_0=0<x_1=x_2=x_3=a.&\end{aligned} \tag{10.3.4}$$

Suppose we are in the first case and $m=1$. The induced module

$$I(-a+1/2,\dots,a-1/2) \tag{10.3.5}$$

is a direct sum of irreducible factors computed in section 10.2, all have lowest K-types of the form μ_e. Consider the module

$$I(a-1/2;\dots;1/2;-a+1/2,\dots,a-1/2). \tag{10.3.6}$$

It is a direct sum of induced modules from the factors of (10.3.5). Each such induced module is a homomorphic image of the corresponding standard module with dominant parameter. So (10.3.6) is also generated by its μ_e isotypic components. But then

$$\begin{aligned}&I(a-1/2;\dots;1/2;-a+1/2,\dots,a-1/2)\cong\\&I(-a+1/2,\dots,a-1/2;a-1/2;\dots;1/2)\end{aligned}\tag{10.3.7}$$

so the latter is also generated by its μ_e isotypic component. Finally, the intertwining operator

$$I(a-1/2;\dots;1/2)\longrightarrow I(1/2,\dots,a-1/2)\tag{10.3.8}$$

is onto, and the results in section 5.3 imply that the image of the intertwining operator

$$I(1/2,\dots,a-1/2)\longrightarrow I(-a+1/2,\dots,-1/2)\tag{10.3.9}$$

is onto $L(-a+1/2,\dots,-1/2)$. Thus

$$I(-a+1/2,\dots,a-1/2;-a+1/2,\dots,-1/2)\tag{10.3.10}$$

is generated by its μ_e isotypic components.

Now suppose that the parameter has another $x_{2m-1}\le x_{2m}$, either case of 10.3.4. The argument above shows that the module

$$I(-x_{2m-1}+1/2,\dots,x_{2m}-1/2,L(\check{\mathcal{O}}_1)).\tag{10.3.11}$$

is generated by its μ_e isotypic components. Precisely, $X(\nu)$ maps onto

$$\begin{aligned}I(x_{2m-1}+1/2,\dots,&x_{2m}-1/2;1/2,\dots,x_0-1/2;\dots;1/2,\dots,x_{2m-2}-1/2;\\&L(-x_{2m-1}+1/2,-x_{2m-1}+1/2,\dots,-1/2,-1/2))\end{aligned}\tag{10.3.12}$$

Replace $L(-x_{2m-1}+1/2,-x_{2m-1}+1/2,\dots,-1/2,-1/2)$ by $I(-x_{2m-1}+1/2,\dots,x_{2m-1}-1/2)$. The ensuing module is a direct sum of induced modules each generated by its μ_e isotypic component. Then observe that the map

$$\begin{aligned}I(x_{2m-1}+1/2,x_{2m}-1/2;&1/2,\dots,x_0-1/2;\dots;1/2,\dots,x_{2m-2}-1/2;\\&-x_{2m-1}+1/2,\dots,x_{2m-1}-1/2)\longrightarrow\\I(-x_{2m-1}+1/2,\dots x_{2m}-1/2;&1/2,\dots,x_0-1/2;\dots;1/2,\dots,x_{2m-2}-1/2)\end{aligned}\tag{10.3.13}$$

is onto. Finally,

$$I(1/2,\dots,x_0-1/2;\dots;1/2,\dots,x_{2m-2}-1/2)\tag{10.3.14}$$

has $L(-x_{2m-2}+1/2,\dots,1/2)$ as its unique irreducible quotient, because it is the homomorphic image of an $X(\nu)$ with ν dominant. Therefore it is generated by its spherical vector.

Thus in the case (10.3.4) with $m>2$, we established that

$$I(-x_{2m-1}+1/2,\dots,x_{2m}-1/2;-a+1/2,\dots,a-1/2;L(\check{\mathcal{O}}_2))\tag{10.3.15}$$

is generated by its μ_e isotypic components. It is isomorphic to

$$I(-a+1/2,\dots,a-1/2,-x_{2m-1}+1/2,\dots,x_{2m}-1/2;L(\check{\mathcal{O}}_2)).\tag{10.3.16}$$

Finally, $I(-x_{2m-1}+1/2,\dots,x_{2m}-1/2;L(\check{\mathcal{O}}_1))$ has a unique irreducible quotient, because it is a homomorphic image of an $X(\nu)$ with ν dominant.

Remains to consider the case when $m=2$ and $x_0=0<x_1=x_2=x_3=a\le x_4$. In this case, the module

$$I(a+1/2,\dots,x_4-1/2;-a+1/2,\dots,a-1/2;-a+1/2,\dots,a-1/2) \quad (10.3.17)$$

is generated by its μ_e isotypic components because of corollary 10.2. Therefore the same holds for

$$I(-a+1/2,\dots,x_4-1/2;-a+1/2,\dots,a-1/2;-a+1/2). \quad (10.3.18)$$

But this is isomorphic to

$$I(-a+1/2,\dots,a-1/2,-a+1/2,\dots,x_4-1/2), \quad (10.3.19)$$

and by section 5.3, $I(-a+1/2,\dots,x_4-1/2)$ has $L(-x_4+1/2,\dots,-1/2,-1/2)$ as a quotient.

References

[ABV] J. Adams, D. Barbasch, D. Vogan, *The Langlands classification and irreducible characters of real reductive groups*, Progress in Mathematics, Birkhäuser, Boston-Basel-Berlin, (1992), vol. 104.

[BB] D. Barbasch, M. Bozicevic *The associated variety of an induced representation*proceedings of the AMS **127 no. 1** (1999), 279-288

[B1] D. Barbasch, *The unitary dual of complex classical groups*, Inv. Math. **96** (1989), 103–176.

[B2] D. Barbasch, *Unipotent representations for real reductive groups*, Proceedings of ICM, Kyoto 1990, Springer-Verlag, The Mathematical Society of Japan, 1990, pp. 769–777.

[B3] D. Barbasch, *The spherical unitary dual for split classical p-adic groups*, Geometry and representation theory of real and p-adic groups (J. Tirao, D. Vogan, and J. Wolf, eds.), Birkhauser-Boston, Boston-Basel-Berlin, 1996, pp. 1–2.

[B4] D. Barbasch, *Orbital integrals of nilpotent orbits* , Proceedings of Symposia in Pure Mathematics, vol. 68, (2000) 97-110.

[B5] D. Barbasch, *The associated variety of a unipotent representation* preprint

[BM] D. Barbasch and A. Moy *A unitarity criterion for p-adic groups*, Inv. Math. **98** (1989), 19–38.

[BM2] ______, *Reduction to real infinitesimal character in affine Hecke algebras*, Journal of the AMS **6 no. 3** (1993), 611-635.

[BM3] ______, *Unitary spherical spectrum for p-adic classical groups*, Acta Applicandae Math **5 no. 1** (1996), 3-37.

[BS] D. Barbasch, M. Sepanski *Closure ordering and the Kostant-Sekiguchi correspondence*, Proceedings of the AMS **126 no. 1** (1998), 311-317.

[BV1] D. Barbasch, D. Vogan *The local structure of characters* J. of Funct. Anal. **37 no. 1** (1980) 27-55

[BV2] D. Barbasch, D. Vogan *Unipotent representations of complex semisimple groups* Ann. of Math., 121, (1985), 41-110

[BV3] D. Barbasch, D. Vogan *Weyl group representation and nilpotent orbits* Representation theory of reductive groups (Park City, Utah, 1982), **Progr. Math., 40**, Birkhuser Boston, Boston, MA, (1983), 21-33.

[CM] D. Collingwood, M. McGovern *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand Reinhold Co., New York, (1993).

[D] D. Djokovic *Closures of conjugacy classes in classical real linear Lie groups II* Trans. Amer. Math. Soc. **270 no. 1**, (1982), 217-252.

[KnV] A. Knapp, D. Vogan *Cohomological induction and unitary representations* Princeton University Press, Princeton Mathematical Series vol. 45, 1995.

[L1] G. Lusztig *Characters of reductive groups over a finite field* Annals of Math. Studies, Princeton University Press vol. 107.

[McG] W. McGovern *Cells of Harish-Chandra modules for real classical groups* Amer. Jour. of Math., 120, (1998), 211-228.

[SV] W. Schmid, K. Vilonen *Characteristic cycles and wave front cycles of representations of reductive groups*, Ann. of Math., 151 (2000), 1071-1118.

[Stein] E. Stein *A*nalysis in matrix space and some new representations of $SL(n, \mathbb{C})$ Ann. of Math. **86** (1967) 461-490

[T] M. Tadic *Classification of unitary representations in irreducible representations of general linear groups*, Ann. Sci. École Norm. Sup. (4) 19, (1986) no. 3, 335-382.

[V1] D. Vogan *The unitary dual of $GL(n)$ over an archimedean field*, Inv. Math., 83 (1986), 449-505.

[V2] ______ *Irreducible characters of semisimple groups IV* Duke Math. J. 49, (1982), 943-1073

[ZE] A. Zelevinsky *Induced representations of reductive $\mathfrak{p}$-adic groups II. On irreducible representations of* $\mathrm{GL}(n)$, Ann. Sci. cole Norm. Sup. (4) **13 no. 2** 165-210

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