Computing Hodge filtrations

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1 Introduction

These notes present an algorithm to compute the Hodge filtration on an arbitrary irreducible representation of a real reductive group. It is based on conversations with Wilfried Schmid and Kari Vilonen. It uses certain properties of the Hodge filtration provided by them, but for which the details have not been written down. We refer to all such statements as conjectures, and have tried to explicitly state what it is that we need. In particular see Conjecture 6.8, Section 7, Conjecture 9.1 and Conjecture 13.11.

The algorithm is very similar to the algorithm of [1] for calculating the signature of the c-form, and ultimately Hermitan forms and the unitary dual. In fact one of the main results of these notes is that the c-form can be thought of as the reduction of the Hodge filtration modulo 2. For a precise statement see Theorem 9.14 and Conjecture 17.6.

The main conjecture of Schmid and Vilonen (Conjecture 7.1) is beyond the scope of this paper. These results do *not* depend on that conjecture. On the other hand the results, including Conjecture 17.6, are consistent with and provide strong support for the conjecture.

Suppose $I(\gamma)$ is an standard $(\mathfrak{g}, K)$-module with real infinitesimal character (we work entirely with representations with real infinitesimal character). It has a canonical c-form and a canonical Hodge filtration. The c-form can be thought of as a function from $\widehat{K}$ to $\mathbb{Z}[s]$ where $s^2 = 1$: the value on a K-type μ is $a + bs$ says that the c-form on this K-isotypic is of signature $a + bs$ (times the positive definite form on μ itself). The c-form is computed by deforming the continuous parameter to 0, and keeping track of the changes to the signature as you cross reducibility points. In this way one obtains a formula for the c-form on $I(\gamma)$ in terms of the c-form on irreducible, tempered representations.

The same idea applies to computing the Hodge filtration on $I(\gamma)$. It is possible to keep track of the changes to the filtration across reducibility points, and this gives a formula for the Hodge filtration on $I(\gamma)$ in terms of those on irreducible, tempered representations. We view the Hodge filtration as a function from $\widehat{K}$ to $\mathbb{Z}[v]$ where v is an indeterminate: the value on a K-type μ is $\sum a_i v^i$ indicates that μ has multiplicity a_i in level i of the Hodge filtration (there is a shift in this indexing, see Definition 4.3).

Our first result is that these two *algorithms* are related, in a precise way, by reduction mod 2. It follows that proving the c-form and the Hodge-filtration are related by reduction mod 2 reduces to the case of tempered representations.

Since a tempered representation is unitary its Hermitian form is positive definite. The algorithm described above computes the c-form on $I(\gamma)$ in terms of those on tempered representations. There is a way to go from the c-form to the ordinary Hermitian form. In the equal rank case this is elementary, although in the unequal rank case it requires a further discussion of the extended group, which we don't dicuss here. In any event this gives a formula for the *Hermitian* form on $I(\gamma)$ in terms of those on irreducible tempered representations. Since tempered representations are unitary these forms are positive definite. This gives an algorithm to compute the Hermitian form on $I(\gamma)$.

In the case of the Hodge filtration there is more work to do: the Hodge filtration on a tempered representation is itself a non-trivial object. If π is an irreducible tempered representation its Hodge filtration is "simple": the lowest K-type is in the lowest degree, and the filtration is obtained from this from the filtration on the universal enveloping algebra, via the action on the lowest K-type. However it is not easy to compute the filtration from this description, and we proceed by an entirely different procedure described in Sections 10–17.

Putting the tempered case and deformation algorithm together we obtain an algorithm to compute the Hodge filtration on $I(\gamma)$, and we see that the c-form and Hodge filtration are related by reduction mod 2.

Finally suppose π is an irreudiclbe $(\mathfrak{g}, K)$-module. Then by the usual Kazhdan-Lusztig-Vogan theory we can write $\pi = \sum a_i I(\gamma_i)$ where the $I(\gamma_i)$ are standard modules. From this we obtain an algorithm for the c-form on π in [1], and a similar argument applies here to compute the c-form.

We have written code in the `atlas` software to compute Hodge filtrations. Some examples are given in Section 19.

We wrote these notes as we were learning about Hodge filtrations, and simultaneously writing code. As a result the notes are a bit disorganized, with some extraneous details in places, and some details missing in others. Nevertheless we hope that they will be helpful in filling in the missing steps and providing guidance for what remains to be done.

2 Multiplicities

We start by recalling a few `atlas` definitions. We're given a connected complex reductive group G, with real points $G(\mathbb{R})$, maximal compact subgroup $K(\mathbb{R})$, with comlexification K. We work entirely in the setting of $(\mathfrak{g}, K)$-modules, always with real infinitesimal character.

We have the notion of a parameter Γ. Various conditions it can satisfy are: standard, final, normal, and non-zero. If Γ is standard, final, and non-zero, then associated to Γ is a standard module $I(\Gamma)$. This has unique irreducible submodule $J(\Gamma)$. (Although `atlas` works with unique irreducible quotients, to be consistent with the Hodge theory literature we prefer to use submodules.) The map from standard, final, non-zero parameters, taking Γ to $J(\Gamma)$ is surjective to the set of irreducible representations. With the appropriate notion of equivalence of parameters this is bijection on the level of equivalence classes.

In this paper we'll use the term *parameter* to refer to a *standard parameter*, and unless otherwise noted all parameters are assumed to be non-zero. In particular a *final* parameter is really a *standard, final, non-zero* parameter.

Sherman, set the Wayback machine to 1980.

For Γ, Ξ final parameters define $m_{\Xi,\Gamma} \in \mathbb{Z}$ by

$$I(\Gamma) = \sum_{\Xi \leq \Gamma} m_{\Xi,\Gamma} J(\Xi)$$

(all such identities are in the Grothendieck group).

Each standard module $X = I(\Gamma)$ comes equipped with its Jantzen filtration, this is a finite increasing filtration

$$0 = JF_{-1}(I(\Gamma)) \subset JF_0(I(\Gamma)) \subset JF_n(I(\Gamma)) \subset JF_{n+1}(I(\Gamma)) = I(\Gamma) \tag{2.1}$$

by $(\mathfrak{g}, K)$-modules. In particular $JF_0(X) = J(X)$ the unique irreducible submodule. Let $\mathrm{gr}JF$ denote the associated graded module:

$$\mathrm{gr}JF_k(X) = JF_k(X)/JF_{k+1}(X) \quad (k = 0, \ldots, n). \tag{2.2}$$

Each $\mathrm{gr}JF_k(X)$ is completely reducible, and $\mathrm{gr}JF_0(I(\Gamma)) = J(\Gamma)$.

We'll also write

$$I(\Gamma, r) = \mathrm{gr}JF_r(I(\Gamma)) = JF_r(I(\Gamma)/JF_{r+1}(I(\Gamma))$$

for the r^{th} graded piece of the Jantzen filtration.

Define $m^r_{\Xi,\Gamma} \in \mathbb{Z}$ by

$$I(\Gamma, r) = \sum_{\Xi \leq \Gamma} m^r_{\Xi,\Gamma} J(\Xi).$$

Thus

$$\sum_{r \geq 0} m^r_{\Xi,\Gamma} = m_{\Xi,\Gamma}$$

Define $Q_{\Xi,\Gamma} \in \mathbb{Z}[q]$ by

$$Q_{\Xi,\Gamma}(q) = \sum_{r \geq 0} m^r_{\Xi,\Gamma} q^{(\ell(\Gamma) - \ell(\Xi) - r)/2} \tag{2.3}$$

In fact $Q^r_{\Xi,\Gamma} \in \mathbb{Z}[q]$, of degree $\leq (\ell(\Gamma) - \ell(\Xi))/2$, and

$$m^r_{\Xi,\Gamma} \neq 0 \Rightarrow \begin{cases} \ell(\Gamma) - \ell(\Xi) \equiv r \\ 0 \leq r \leq \ell(\Gamma) - \ell(\Xi) \end{cases} \tag{2.4}$$

(we write $\equiv$ for equivalence $\pmod 2$). Thus we can write (2.3) more precisely as

$$\begin{aligned} Q_{\Xi,\Gamma}(q) &= \sum_{k=0}^{(\ell(\Gamma)-\ell(\Xi))/2} m^{\ell(\Gamma)-\ell(\Xi)-2k}_{\Xi,\Gamma} q^k \\ &= \sum_{\substack{r=0 \\ r \equiv \ell(\Gamma)-\ell(\Xi)}}^{\ell(\Gamma)-\ell(\xi)} m^r_{\Xi,\Gamma} q^{(\ell(\Gamma)-\ell(\Xi)-r)/2} \end{aligned} \tag{2.5}$$

3 Filtrations, Gradings and Functions

This section contains some formalism about filtrations, gradings and functions.

Suppose π is a K-module (for example the restriction of a $(\mathfrak{g}, K)$-module), equipped with a K-invariant grading $\pi = \sum_i \mathrm{gr}_i(\pi)$ (for example the Hodge grading). We define the associated *grading function* to be $f_\pi : \widehat{K} \to \mathbb{N}[v]$ defined by

$$f_\pi(\mu) = \sum_i \mathrm{mult}(\mu, \mathrm{gr}_{i+c}(\pi))v^i.$$

Here c is an optional degree shift (often the codimension of a K-orbit). It is convenient to write this

$$\mathrm{gr}(\pi)|_K = \sum_\mu f_\pi(\mu)\mu.$$

If π is a virtual K-module the same holds with $\mathbb{N}[v]$ replaced by $\mathbb{Z}[v]$.

We frequently have the following situation. We're given two representations of K, π and σ, with σ finite-dimensional, and each equipped with a grading. Then $\pi \otimes \sigma$ has a natural grading satisfying:

(3.1)
$$\mathrm{gr}_n(\pi \otimes \sigma)|_K = \sum_{p+q=n} \mathrm{gr}_p(\pi) \otimes \mathrm{gr}_q(\sigma)$$

In other words if we have

$$\mathrm{gr}(\pi)|_K = \sum_\mu f_\pi(\mu)\mu, \quad \mathrm{gr}(\sigma)|_K = \sum_\mu f_\sigma(\mu)\mu$$

then

(3.2)(a)
$$\mathrm{gr}(\pi \otimes \sigma)|_K = \sum_{\phi,\psi} f_\mu(\phi) f_\sigma(\psi) \phi \otimes \psi$$

In terms of grading functions, write $f_\pi \otimes f_\sigma$ for the grading function of the grading (a). Then

(3.2)(b)
$$(f_\pi \otimes f_\sigma)(\mu) = \sum_{\phi,\psi} \mathrm{mult}(\mu, \phi \otimes \psi) f_\pi(\phi) f_\sigma(\psi)$$

Suppose π is an irreducible or standard module. It has a canonical (K-invariant) *Hodge filtration* $\{\mathcal{F}_p(\pi)\}$, with K-invariant associated grading $\mathrm{gr}(\pi)$. As noted above we define the *Hodge function* of π to be the associated function, with degree shift by $a(\pi)$, the codimension of the underlying K-orbit:

$$\mathrm{hodge}(\pi)(\mu) = \sum \mathrm{mult}(\mu, \mathrm{gr}_{i+a(\pi)})v^i.$$

Note that the Hodge grading $\mathrm{gr}(\pi)$ does not determine the Hodge filtration $\mathcal{F}_p(\pi)$, and $\mathrm{hodge}(\pi)$ only determines $\mathrm{gr}(\pi)$ as a K-module.

3.1 The Graded Koszul identity

Suppose V is a finite dimensional representation of K. Define $\mathcal{S}^\bullet(V)$ to be the symmetric algebra of V equipped with the grading by degree. Let symm be the function of this grading, i.e.

(3.1.3)(a)
$$\mathrm{symm}(V) = \sum_k \mathrm{mult}(\mathcal{S}^k(V))v^k.$$

This is a function from $\widehat{K}$ to $\mathbb{N}[v]$:

(3.1.3)(b)
$$\mathrm{symm}(V)(\mu) = \sum_k \mathrm{mult}(\mu, \mathcal{S}^k(V))v^k \quad (\mu \in \widehat{K}).$$

Another version is

(3.1.3)(c)
$$\mathcal{S}^\bullet(V) = \sum_{\mu \in \widehat{K}} \mathrm{symm}(V)(\mu)\mu$$

Similarly define $\bigwedge^\bullet(\mathfrak{u}\cap\mathfrak{s})$ to be the exterior algebra, graded by (alternating) degree. The grading function is

$$\mathrm{alt}(\mathfrak{u}\cap\mathfrak{s}) = \sum_k \mathrm{mult}(\bigwedge^k(\mathfrak{u}\cap\mathfrak{s}))(-v)^k,$$

which is shorthand for

$$\mathrm{alt}(\mathfrak{u}\cap\mathfrak{s})(\mu) = \sum_k \mathrm{mult}(\mu, \bigwedge^k(\mathfrak{u}\cap\mathfrak{s}))(-v)^k.$$

or equivalently

$$\bigwedge^{\bullet}(\mathfrak{u}\cap\mathfrak{s})=\sum_{\mu\in\widehat{K}}\mathrm{alt}(\mathfrak{u}\cap\mathfrak{s})(\mu)\mu$$

Now suppose $\mathfrak{q}=l\oplus\mathfrak{u}$ is a θ-stable parabolic subalgebra. The Koszul complex $\mathfrak{q}$ is

$$\mathbb{C}_L=\mathcal{S}^{\bullet}(\mathfrak{u}\cap\mathfrak{s})\otimes\bigwedge^{\bullet}(\mathfrak{u}\cap\mathfrak{s}) \tag{3.1.4}$$

This is a graded identity, so we have (using (3.1)):

$$\mathrm{hodge}(\mathbb{C}_L)=\mathrm{symm}(\mathfrak{u}\cap\mathfrak{s})\otimes\mathrm{alt}(\mathfrak{u}\cap\mathfrak{s}) \tag{3.1.5}$$

I find this a bit too terse, and find it helpful to recall this means

$$\mathrm{hodge}(\mathbb{C}_L)(\mu)=\sum_{p,q}\mathrm{mult}(\mu,S^p(\mathfrak{u}\cap\mathfrak{s})\otimes\bigwedge^{q}(\mathfrak{u}\cap\mathfrak{s}))v^p(-v)^q$$

4 Hodge Filtration

Definition 4.1 *Suppose Γ is a (standard) parameter, but not necessarily final. If the infinitesimal character of Γ is regular then Γ is associated to a unique K-orbit $\mathcal{O}$ on the flag variety, and we set*

$$a(\Gamma)=\mathrm{codim}(\mathcal{O})$$
$$b(\Gamma)=\dim(\mathcal{O})$$

If Γ is singular then Γ may be associated to several orbits $\mathcal{O}_1,\dots,\mathcal{O}_n$.

Assume Γ is final (meaning non-zero), and define $a(\Gamma)=\max_i(codim(\mathcal{O}_i))$, $b(\Gamma)=\min_i(\dim(\mathcal{O}_i))$.

In `atlas` terms, if $\Gamma=(x,\lambda,\nu)$ is a standard, final, limit parameter (with emphasis on *final*) then

$$b(\Gamma)=\dim(x);$$

atlas automatically (in `finalize`) moves to the smallest orbit.

Suppose Γ is a parameter and $X=I(\Gamma)$ (a standard module) or $J(\Gamma)$ (an irreducible module). Then X comes equipped with a canonical increasing *Hodge filtration*:

$$0=HF_{a-1}(X)\subset HF_a(X)\subset HF_{a+1}(X)\subset\dots. \tag{4.2}$$

where $a = a(\Gamma)$. Each $HF_k(X)$ is a K-module. I think if Γ is final the lowest K-types of X are always in $HF_a(X)$; in particular $HF_a(X) \neq 0$.

We set $\mathrm{grHF}_k(X) = HF_k(X)/HF_{k-1}(X)$ $(k \geq a)$. Each grHF_k is a representation of K. We call this the k^{th} level of the Hodge grading.

We define the *Hodge filtration* function by analogy with sig. As in that case we normalize it to have nonzero constant term. Suppose Γ is a (non-zero) standard final parameter, and $X = I(\Gamma)$ or $J(\Gamma)$. Informally we write

$$\mathrm{hodge}(X) = \sum_{\mu \in \widehat{K}} f_\mu(v)\mu$$

where $f_\mu(v) \in \mathbb{Z}[v]$; the coefficient of v^k is the multiplicity of μ in $\mathrm{grHF}_{a(\Gamma)+k}(X)$.

Definition 4.3 *Suppose Γ is a (non-zero) standard final parameter, and $X = I(\Gamma)$ or $J(\Gamma)$. Define the Hodge function of X to be the function hodge(X) from $\widehat{K}$ to $\mathbb{Z}[v]$ defined as follows. If $\mu \in \widehat{K}$ then*

$$\mathrm{hodge}(X)(\mu) = \sum_{k=0}^{\infty} c_k v^k \quad (c_k = \mathrm{mult}(\mu, \mathrm{grHF}_{a(\Gamma)+k}(X))$$

Alternatively we'll write $I_v(\Gamma) = \mathrm{hodge}(I(\Gamma))$ and $J_v(\Gamma) = \mathrm{hodge}(J(\Gamma))$.

Note that we've normalized so that if μ is a lowest K-type of $I(\Gamma)$ then $\mathrm{hodge}(I(\Gamma))(\mu) = 1$ (rather than $v^{a(\Gamma)}$).

For $\mu \in \widehat{K}$, $\mathrm{hodge}(X)(\mu)$ is a polynomial satisfying

$$\mathrm{hodge}(X)(\mu)(1) = \mathrm{mult}(\mu, X)$$

Thus

$$I_v(\Gamma)(\mu) \in \mathbb{Z}[v] \text{ for all } \mu \tag{4.4)(a}$$

and if μ is a lowest K-type then

$$I_v(\Gamma)(\mu) = 1. \tag{4.4)(b}$$

The same holds with J in place of I.

Each standard module $X = I(\Gamma)$ comes equipped with its canonical finite weight filtration coming from Hodge theory. We assume this has been chosen to be a *decreasing* filtration (with $b = b(\Gamma)$):

$$X = W_b(X) \supset W_{b+1}(X) \supset \cdots \supset W_n(X) \supset W_{n+1}(X) = 0 \tag{4.5}$$

by $(\mathfrak{g}, K)$-modules. This is equal to the Jantzen filtration (Section 2) up to a shift. If J is irreducible it has the trivial weight filtration $W_b(J) = J$.

Let grW denote the associated graded module:

$$\mathrm{grW}_k(X) = W_k(X)/W_{k+1}(X) \quad (k = b, \ldots, n). \tag{4.6}$$

In particular $\mathrm{grW}_b(I(\Gamma)) = J(\Gamma)$.

Desideratum 4.7 *Suppose Γ is a standard final limit parameter. The Hodge and weight filtrations are normalized with the Hodge filtration starting in degree $a(\Gamma)$ and the weight filtration in degree $b(\Gamma)$. The lowest K-types are in $HF_a(I(\Gamma))$; in particular $HF_a(I(\Gamma)) \neq 0$.*

The Hodge filtration of $I(\Gamma)$ induces a filtration on each graded piece $\mathrm{grW}_r(I(\Gamma))$ of the weight filtration.

Definition 4.8 *Define the Hodge filtration function on the r^{th} graded piece of the Jantzen filtration by:*

$$\mathrm{hodge}(I(\Gamma), r)(\mu) = \sum_k c_k v^k \quad \text{where } c_k = \mathrm{mult}(\mu, \mathrm{grHF}_{a(\Gamma)+k}(\mathrm{grW}_r I(\Gamma)))$$

5 Functoriality

See [3].

Suppose $\phi\colon X \to Y$ is a morphism of $(\mathfrak{g}, K)$-modules. Assume X, Y come equipped with Hodge and weight filtrations, and that ϕ is *functorially constructible* (I think this means coming from a morphism of sheaves).

Proposition 5.1 *Both the Hodge and weight filtrations are strictly preserved by functorially constructible morphisms:*

(a) $\phi(HF_k(X)) = (\phi(X)) \cap HF_k(Y)$

(b) $\phi(W_k(X)) = (\phi(X)) \cap W_k(Y)$

Example 5.2 For $SL(2,\mathbb{R})$, the injection of a limit of discrete series into a reducible principal series is *not* functorially constructible. In the notation of [2] there is no homomorphism from $\mathcal{M}_{\{0\},0}$ to $\mathcal{M}_{\mathbb{C}^*,0,\mathrm{odd}}$. The latter has a unique submodule which has no global sections (in the case of regular infinitesimal character the global sections of this submodule is the finite dimensional submodule of the induced representation).

Suppose X is a $(\mathfrak{g},K)$-module equipped with a filtration $HF_*(X)$.

If $\phi\colon Y \hookrightarrow X$ is a $(\mathfrak{g},K)$-module injection then F induces a filtration on Y by: $\phi(HF_k(Y)) = HF_k(X)\cap\phi(Y)$.

Similarly if $\phi\colon X \twoheadrightarrow Y$ is a surjection, then we obtain a filtration on Y by $HF_k(Y)=\phi(HF_k(X))$.

In particular suppose $I=I(\Gamma)$ is a standard (final, limit) module, equipped with its canonical Hodge filtration HF_*. This induces a filtration on each summand $W_j(X)$, each graded module $\mathcal{W}_j(X)$, and each irreducible summand of $\mathcal{W}_j(X)$.

Desideratum 5.3 *Suppose Γ,Ξ are a standard final limit parameters, and π is an irreducible submodule of $grW_j(I(\Gamma))$. Then the filtration on π, induced by the canonical Hodge filtration of $I(\Gamma)$, depends only on the equivalence class of π. Furthermore this filtration differs from the canonical Hodge filtration of π by a shift.*

This is supposed to be a consequence of functoriality.

6 Some more formalism

We need the analogue of $w^{c,r}_{\Xi,\Gamma}$ and $Q^c_{\Xi,\Gamma}$ (see Section 8).

So suppose Γ_t is a family of parameters which has an isolated reducibility point at $t=1$ and set $\Gamma=\Gamma_1$. For t generic $I(\Gamma_t)$ is irreducible, and we write $\mathrm{hodge}(I(\Gamma_t))=\mathrm{hodge}(J(\Gamma_t))$ accordingly.

Recall we've incorporated the shift by $-a(\Gamma)$ into the definition of the Hodge function.

Definition 6.1 *Define $w^H_{\Xi,\Gamma}\in\mathbb{Z}[v]$ by:*

$$\mathrm{hodge}(I(\Gamma))=\sum_{\Xi\le\Gamma} w^H_{\Xi,\Gamma}\,\mathrm{hodge}(J(\Xi)). \tag{6.2}$$

Recall (Definition 4.8) the functions $\mathrm{hodge}(I(\Gamma), r)$*. Define* $w^{H,r}_{\Xi,\Gamma} \in \mathbb{Z}[v]$ *by:*

$$\mathrm{hodge}(I(\Gamma), r) = \sum_{\Xi \leq \Gamma} w^{H,r}_{\Xi,\Gamma} \, \mathrm{hodge}(J(\Xi)).$$

By analogy with [1, Definition 20.2] *define* $Q^H_{\Xi,\Gamma} \in \mathbb{Z}[v, q]$ *by:*

$$Q^H_{\Xi,\Gamma}(q) = \sum_{r \geq 0} w^{H,r}_{\Xi,\Gamma} q^{(\ell(\Gamma) - \ell(\Xi) - r)/2} \tag{6.3}$$

Lemma 6.4

(1) $w^c_{\Xi,\Gamma} = \sum_r w^{c,r}_{\Xi,\Gamma}$

(2) $w^H_{\Xi,\Gamma} = \sum_r w^{H,r}_{\Xi,\Gamma}$

(3) $w^{c,r}_{\Xi,\Gamma}(s = 1) = m^r_{\Xi,\Gamma}$

(4) $w^c_{\Xi,\Gamma}(s = 1) = m_{\Xi,\Gamma}$

(5) $w^{H,r}_{\Xi,\Gamma}(v = 1) = m^r_{\Xi,\Gamma}$

(6) $w^H_{\Xi,\Gamma}(v = 1) = m_{\Xi,\Gamma}$

(7) $w^{c,r}_{\Xi,\Gamma}$ *is a pure element of* $\mathbb{W} = \mathbb{Z}[s]$*, i.e. in* $\mathbb{Z} \cup \mathbb{Z}s$*.*

(8) Assuming Desideratum 5.3 $w^{H,r}_{\Xi,\Gamma} = m^r_{\Xi,\Gamma} v^{c(\Xi,\Gamma,r)}$ *for some non-negative integer* $c(\Xi, \Gamma, r)$*.*

(9) $Q^c_{\Xi,\Gamma}(s = 1, q = 1) = Q_{\Xi,\Gamma}(q = 1) = m_{\Xi,\Gamma}$

(10) $Q^H_{\Xi,\Gamma}(v = 1, q = 1) = Q_{\Xi,\Gamma}(q = 1) = m_{\Xi,\Gamma}$

(11) $Q^H_{\Xi,\Gamma}(q = 1) = w^H_{\Xi,\Gamma}$

(12) $\mathrm{hodge}(I(\Gamma)) = \sum_{\Xi \leq \Gamma} Q^H_{\Xi,\Gamma}(q = 1) \mathrm{hodge}(J(\Xi))$

Statements (1), (3), (4) and (7) (9) are (easy, and) in [1], and (2), (5), (6) and (10) are the immediate analogues for $w^H_{\Xi,\Gamma}$ and $w^{H,r}_{\Xi,\Gamma}$. Part (8) is simply a reformulation of the Desideratum, (11) is immediate from the definitions, and (11) implies (12).

I find it helpful to recall some parallel definitions. Here are the defining properties of $m_{\Xi,\Gamma}, w^c_{\Xi,\Gamma}$ and $w^H_{\Xi,\Gamma}$.

$$
(6.5)\qquad \begin{aligned}
I(\Gamma) &= \sum_{\Xi\le\Gamma} m_{\Xi,\Gamma} J(\Xi) \quad (m_{\Xi,\Gamma}\in\mathbb{Z})\\
\mathrm{sig}(\mathrm{gr}(I(\Gamma))) &= \sum_{\Xi\le\Gamma} w^c_{\Xi,\Gamma}\,\mathrm{sig}(J(\Xi)) \quad (w^c_{\Xi,\Gamma}\in\mathbb{Z}[s])\\
\mathrm{hodge}(I(\Gamma)) &= \sum_{\Xi\le\Gamma} w^H_{\Xi,\Gamma}\,\mathrm{hodge}(J(\Xi)) \quad (w^H_{\Xi,\Gamma}\in\mathbb{Z}[v])
\end{aligned}
$$

Here are the defining properties of of $m^r_{\Xi,\Gamma}, w^{c,r}_{\Xi,\Gamma}$ and $w^{H,r}_{\Xi,\Gamma}$.

$$
(6.6)\qquad \begin{aligned}
I(\Gamma,r) &= \sum_{\Xi\le\Gamma} m^r_{\Xi,\Gamma} J(\Xi) \quad (m^r_{\Xi,\Gamma}\in\mathbb{Z})\\
\mathrm{sig}(I(\Gamma),r) &= \sum_{\Xi\le\Gamma} w^{c,r}_{\Xi,\Gamma}\,\mathrm{sig}(J(\Xi)) \quad (w^{c,r}_{\Xi,\Gamma}\in\mathbb{Z}[s])\\
\mathrm{hodge}(I(\Gamma),r) &= \sum_{\Xi\le\Gamma} w^{H,r}_{\Xi,\Gamma}\,\mathrm{hodge}(J(\Xi)) \quad (w^{H,r}_{\Xi,\Gamma}\in\mathbb{Z}[v])
\end{aligned}
$$

Here are the definitions of Q, Q^c and Q^H:

$$
(6.7)\qquad \begin{aligned}
Q_{\Xi,\Gamma}(q) &= \sum_{r\ge 0} m^r_{\Xi,\Gamma} q^{(\ell(\Gamma)-\ell(\Xi)-r)/2} \quad (Q_{\Xi,\Gamma}\in\mathbb{Z}[q])\\
Q^c_{\Xi,\Gamma}(q) &= \sum_{r\ge 0} w^{c,r}_{\Xi,\Gamma} q^{(\ell(\Gamma)-\ell(\Xi)-r)/2} \quad (Q^c_{\Xi,\Gamma}\in\mathbb{Z}[s,q])\\
Q^H_{\Xi,\Gamma}(q) &= \sum_{r\ge 0} w^{H,r}_{\Xi,\Gamma} q^{(\ell(\Gamma)-\ell(\Xi)-r)/2} \quad (Q^H_{\Xi,\Gamma}\in\mathbb{Z}[v,q])
\end{aligned}
$$

Recall $w^{c,r}_{\Xi,\Gamma}\in\mathbb{Z}\cup\mathbb{Z}s$ (Lemma 6.4(7)), and more precisely (Proposition 8.11):

$$
\begin{aligned}
Q^c_{\Xi,\Gamma}(s,q) &= s^{(\ell_0(\Gamma)-\ell_0(\Xi))/2} Q_{\Xi,\Gamma}(sq)\\
w^{c,r}_{\Xi,\Gamma} &= s^{(\ell_0(\Gamma)-\ell_0(\Xi))/2} s^{(\ell(\Gamma)-\ell(\Xi)-r)/2} m^r_{\Xi,\Gamma}
\end{aligned}
$$

Here is the analogue in the Hodge filtration setting. I think this is a consequence of strong functoriality (Section 5).

Conjecture 6.8

(6.9)(a) $\qquad w^{H,r}_{\Xi,\Gamma}\in\mathbb{Z}v^k$ *(for some $k\in\mathbb{Z}\ge 0$),*

$$Q^H_{\Xi,\Gamma} = v^{(\ell_0(\Gamma)-\ell_0(\Xi))/2} Q_{\Xi,\Gamma}(vq) \tag{6.9)(b}$$

and

$$w^{H,r}_{\Xi,\Gamma} = v^{(\ell_0(\Gamma)-\ell_0(\Xi))/2} v^{(\ell(\Gamma)-\ell(\Xi)-r)/2} m^r_{\Xi,\Gamma} \tag{6.9)(c}$$

It is easy to see that (b)$\Leftrightarrow$(c)$\Rightarrow$(a).

Remark 6.10 Here is how I think of this; see Remark 8.15.

Assume the infinitesimal character is integral, so all orientation numbers are 0. The "default" level of the Jantzen filtration for $J(\Xi)$ to occur in is $\ell(\Gamma)-\ell(\Xi)$. If $J(\Xi)$ occurs in that level, the restriction of the Hodge filtration function of $I(\Gamma)$ to $J(\Xi)$, and the canonical Hodge filtration function of $J(\Xi)$, agree. If it occurs instead in this shifted by $2k$, then there is a shift by v^k. If the infinitesimal character isn't integral the same holds, except there is also an orientation number term.

Lemma 6.11

(1) $w^H_{\Xi,\Gamma}(v=1) = m^H_{\Xi,\Gamma}$

(2) $w^{H,r}_{\Xi,\Gamma}(v=1) = m^r_{\Xi,\Gamma}$

(3) $Q^H_{\Xi,\Gamma}(v=1,q) = Q_{\Xi,\Gamma}$

Assume Conjecture 6.8. Then:

(5) $w^{H,r}_{\Xi,\Gamma}(v=s) = w^{c,r}_{\Xi,\Gamma}$

(6) $Q^H_{\Xi,\Gamma}(v=s,q) = Q^c_{\Xi,\Gamma}$

Proof. We already have (1-3) (Lemma 6.4). Assuming the conjecture we have

$$w^{H,r}_{\Xi,\Gamma}(v=s) = s^{\ell_0(\Gamma)-\ell_0(\Xi)} s^{(\ell(\Gamma)-\ell(\Xi)-r)/2} m^r_{\Xi,\Gamma}$$

and by Proposition 8.11

$$w^{c,r}_{\Xi,\Gamma} = s^{\ell_0(\Gamma)-\ell_0(\Xi)} s^{(\ell(\Gamma)-\ell(\Xi)-r)/2} m^r_{\Xi,\Gamma}.$$

Then (6) follows from (5) and the definitions of $Q^H_{\Xi,\Gamma}$ (6.9)(b) and $Q^c_{\Xi,\Gamma}$ (8.5). $\square$

7 The Schmid-Vilonen Conjecture

We state the main Conjecture of Schmid and Vilonen in three parts.

Conjecture 7.1 *Suppose X is irreducible.*

(1) The restriction of the c-form to $HF_k(X)$ is non-degenerate for all k.

(2) The restriction of the c-form to $HF_k(X) \cap HF_{k+1}(X)^\perp$ is definite.

(3) If $v \in HF_k(X) \cap HF_{k+1}(X)^\perp$ then $(-1)^{a+p}(v, v) > 0$.

See [3, Conjecture 1.10].

Here is a related conjecture.

Conjecture 7.2 *Suppose Γ is a final parameter. The Hodge filtration function and the signature function satisfy*

(a) $\mathrm{hodge}(J(\Gamma))|_{v=s} = \mathrm{sig}(J(\Gamma))$

(b) $\mathrm{hodge}(I(\Gamma), r)|_{v=s} = \mathrm{sig}(I(\Gamma), r)$

(c) $\mathrm{hodge}(I(\Gamma))|_{v=s} = \mathrm{sig}(\mathrm{gr}(I(\Gamma)))$

Recall we've normalized the signature to be positive on the lowest K-types, and the Hodge filtration function to be 1 there. So these formulas hold without any extra power of s.

I believe Conjecture 7.1 implies Conjecture 7.2 (but not vice versa). I also believe:

(1) Assuming Conjecture 6.8 we have (a)$\Rightarrow$(b)$\Rightarrow$(c)

(2) Assuming Conjectures 6.8 and 9.1, the algorithm in Section 9 implies it is enough to prove (1) for all tempered parameters Γ

Here is the main conclusion of this. Assuming the details in the preceding sketch can be filled in, then we can prove:

Conjecture 7.3 *Assume conjectures 6.8 and 9.1. Then (1) implies (3) in Conjecture 7.1.*

8 Deformation calculation of the c-form

We start by recalling the deformation calculation of the c-form, filling in a few details, in preparation for a similar calculation of Hodge filtrations. We follow [1, Section 21] We follow the notation from *loc. cit.*, simplified a bit in places.

We recall the definition of the signature character. Suppose X is a $(\mathfrak{g}, K)$-module with real infinitesimal character, so it has a c-invariant form (canonical if X is irreducible). Informally we write

$$\mathrm{sig}(X) = \sum_{\mu \in \widehat{K}} w_\mu \mu$$

where $\widehat{K}$ is the set of irreducible representations of K, and $w_\mu = a + bs \in \mathbb{W} = \mathbb{Z}[s]$ ($s^2 = 1$) indicates the signature of the c-form on the μ-isotypic subspace of X. (We work exclusively with the c-invariant form, so there is no danger of confusion with the invariant Hermitian form.)

More precisely $\mathrm{sig}(X)$ is the map from $\widehat{K}$ to $\mathbb{W}$ defined as follows. Write the μ-isotypic subspace $X[\mu]$ of X as

$$X[\mu] = \mathrm{Hom}_K(\mu, X) \otimes \mu.$$

The restriction of the c-invariant form to $X[\mu]$ induces a form on $\mathrm{Hom}_K(\mu, X)$, a vector space of dimension the multiplicity of μ, nondegenerate if X is irreducible. Suppose (say X is irreducible) this space has signature (a, b). Then $\mathrm{sig}(X)(\mu) = a + bs$.

In particular if X is an irreducible or standard module and μ is a lowest K-type of X then

$$\mathrm{sig}(X)(\mu) = 1. \tag{8.1}$$

Now suppose Γ is a parameter. Write

$$\mathrm{sig}(I(\Gamma), r)$$

for the signature of the c-invariant form on $I(\Gamma, r)$: this is the c-invariant form on the r^{th} graded level of the Jantzen filtration, induced by the c-invariant form on $I(\Gamma)$ (by taking limits).

Now suppose Γ_t is a family of parameters which is reducible at $t = 1$, irreducible for $0 < |t| < \epsilon$, and set $\Gamma = \Gamma_1$ (since our modules have real infinitesimal character, $t \in \mathbb{R}$). For t generic $I(\Gamma_t)$ is irreducible, and we write $\mathrm{sig}(I(\Gamma_t)) = \mathrm{sig}(J(\Gamma_t))$ accordingly.

Define $w^c_{\Xi,\Gamma} \in \mathbb{W}$ by the equality:

$$\mathrm{sig}(\mathrm{gr}(I(\Gamma))) = \sum_{\Xi} w^c_{\Xi,\Gamma}\mathrm{sig}(J(\Xi)). \tag{8.2}$$

In other words

$$w^c_{\Xi,\Gamma} = \mathrm{mult}(\mathrm{sig}(J(\Xi)) \text{ in } \mathrm{sig}(\mathrm{gr}(I(\Gamma)))$$

We've simplified the notation slightly from [1, (15.11)(c)].

Also define $w^{c,r}_{\Xi,\Gamma}$ by:

$$\mathrm{sig}(I(\Gamma), r) = \sum_{\Xi\le\Gamma} w^{c,r}_{\Xi,\Gamma}\mathrm{sig}(J(\Xi)).$$

i.e.

$$w^{c,r}_{\Xi,\Gamma} = \mathrm{mult}(\mathrm{sig}(J(\Xi)) \text{ in } \mathrm{sig}(I(\Gamma), r)$$

Therefore

$$w^c_{\Xi,\Gamma} = \sum_{r\ge 0} w^{c,r}_{\Xi,\Gamma} \tag{8.3}$$

Also

$$\begin{aligned} w^{c,r}_{\Xi,\Gamma}(s=1) &= m^r_{\Xi,\Gamma} \\ w^c_{\Xi,\Gamma}(s=1) &= m_{\Xi,\Gamma} \end{aligned} \tag{8.4}$$

Following [1, Definition 20.2] define $Q^c_{\Xi,\Gamma} \in \mathbb{W}[q]$ by:

$$Q^c_{\Xi,\Gamma}(q) = \sum_{r\ge 0} w^{c,r}_{\Xi,\Gamma} q^{(\ell(\Gamma)-\ell(\Xi)-r)/2} \tag{8.5}$$

Since $\mathbb{W} = \mathbb{Z}[s]$ it is sometimes useful to write $Q^c_{\Xi,\Gamma} \in \mathbb{Z}[s,q]$, especially when specializing q or s.

By $Q^c_{\Xi,\Gamma}(1)$ we always mean $Q^c_{\Xi,\Gamma}(s, q = 1) \in \mathbb{Z}[s]$. We have

$$\begin{aligned} Q^c_{\Xi,\Gamma}(1) &= w^c_{\Xi,\Gamma} \\ \mathrm{sig}(\mathrm{gr}(I(\Gamma))) &= \sum_{\Xi} Q^c_{\Xi,\Gamma}(1)\mathrm{sig}(J(\Xi)). \end{aligned} \tag{8.6}$$

In fact this is in $\mathbb{W}[q]$, of degree $\le (\ell(\Gamma)-\ell(\Xi))/2$. In particular

$$w^r_{\Xi,\Gamma}\ne 0 \Rightarrow \begin{cases}\ell(\Gamma)-\ell(\Xi)\equiv r\\ 0\le r\le \ell(\Gamma)-\ell(\Xi)\end{cases} \tag{8.7}$$

It is possible to say more but this is all we need. See [1, Proposition 20.3]. The analogue of (2.5) is

$$Q^c_{\Xi,\Gamma}(q)=\sum_{k=0}^{(\ell(\Gamma)-\ell(\Xi))/2}\mathrm{mult}(\mathrm{sig}(J(\Xi))\text{ in }\mathrm{sig}(I(\Gamma,\ell(\Gamma)-\ell(\Xi)-2k)))q^k$$

Note that

$$\begin{aligned}Q^c_{\Xi,\Gamma}(s=1,q)&=Q_{\Xi,\Gamma}\in\mathbb{Z}[q]\\ Q^c_{\Xi,\Gamma}(s,q=1)&=w^c_{\Xi,\Gamma}\in\mathbb{Z}[s]\\ Q^c_{\Xi,\Gamma}(s=1,q=1)&=m_{\Xi,\Gamma}\in\mathbb{Z}\end{aligned} \tag{8.8}$$

The next Lemma is [1, Corollary 15.12] (which is older), written in a form which lends itself to generalizing to the case of Hodge filtrations.

Lemma 8.9

$$\sum_{r\ge 0}(1-s^r)\mathrm{sig}(I(\Gamma,r))=\sum_{\Xi<\Gamma}(1-s^{\ell(\Gamma)-\ell(\Xi)})Q^c_{\Xi,\Gamma}(q=1)\mathrm{sig}(J(\Xi))$$

Since $s^2=1$ this simplifies considerably to:

$$(1-s)\sum_{r\text{ odd}}\mathrm{sig}(I(\Gamma,r))=(1-s)\sum_{\substack{\Xi<\Gamma\\ \ell(\Gamma)-\ell(\Xi)\text{ odd}}}Q^c_{\Xi,\Gamma}(q=1)\mathrm{sig}(J(\Xi)) \tag{8.10}$$

which is [1, Corollary 15.12].

Proof.

$$\begin{aligned}
\sum_{r\ge 0}(1-s^r)\mathrm{sig}(I(\Gamma,r)) &= \sum_{r\ge 0}(1-s^r)\sum_{\substack{\Xi<\Gamma\\ \ell(\Gamma)-\ell(\Xi)=r}} w^{c,r}_{\Xi,\Gamma}\mathrm{sig}(J(\Xi))\\
&= \sum_{\Xi<\Gamma}\Big\{\sum_{\substack{r\\ r\equiv\ell(\Gamma)-\ell(\Xi)}}(1-s^r)w^{c,r}_{\Xi,\Gamma}\Big\}\mathrm{sig}(J(\Xi))\\
&= \sum_{\Xi<\Gamma}\Big\{\sum_{\substack{r\\ r\equiv\ell(\Gamma)-\ell(\Xi)}} w^{c,r}_{\Xi,\Gamma} - \sum_{\substack{r\\ r\equiv\ell(\Gamma)-\ell(\Xi)}} s^r w^{c,r}_{\Xi,\Gamma}\Big\}\mathrm{sig}(J(\Xi))\\
&= \sum_{\Xi<\Gamma}\Big\{\sum_{\substack{r\\ r\equiv\ell(\Gamma)-\ell(\Xi)}} w^{c,r}_{\Xi,\Gamma} - s^{\ell(\Gamma)-\ell(\Xi)}\sum_{\substack{r\\ r\equiv\ell(\Gamma)-\ell(\Xi)}} w^{c,r}_{\Xi,\Gamma}\Big\}\mathrm{sig}(J(\Xi))\\
&= \sum_{\Xi<\Gamma}(1-s^{\ell(\Gamma)-\ell(\Xi)})\Big\{\sum_{\substack{r\\ r\equiv\ell(\Gamma)-\ell(\Xi)}} w^{c,r}_{\Xi,\Gamma}\Big\}\mathrm{sig}(J(\Xi))\\
&= \sum_{\Xi<\Gamma}(1-s^{\ell(\Gamma)-\ell(\Xi)})Q^c_{\Xi,\Gamma}(1)\mathrm{sig}(J(\Xi))
\end{aligned}$$

□

One of the main results of [1] (Theorem 20.6) is:

Proposition 8.11

$$Q^c_{\Xi,\Gamma}(s,q) = s^{(\ell_0(\Gamma)-\ell_0(\Xi))/2}Q_{\Xi,\Gamma}(sq) \tag{8.12}$$

Equivalently:

$$w^{c,r}_{\Xi,\Gamma} = s^{(\ell_0(\Gamma)-\ell_0(\Xi))/2}s^{(\ell(\Gamma)-\ell(\Xi)-r)/2}m^r_{\Xi,\Gamma} \tag{8.13}$$

The second version is equivalent to the first, using the definitions of $m^r_{\Xi,\Gamma}$ (2.3) and $w^{c,r}_{\Xi,\Gamma}$ (8.5).

Remark 8.14 According to [1, Theorem 20.6]

$$Q^c_{\Xi,\Gamma}(s,q) = s^{(\ell_0(\Xi)-\ell_0(\Gamma))/2}Q_{\Xi,\Gamma}(sq).$$

This differs from (8.12) in the sign of the exponent of s, which is immaterial since $s^2=1$. However in the Hodge filtration setting this difference matters. See Section 9.

This is equivalent to ([1, Theorem 20.6(2b)]): if $J(\Xi)$ occurs in $\mathrm{gr} JF_r(I(\Gamma))$ then

$$\mathrm{sig}(I(\Gamma), r) = s^{(\ell(\Gamma)-\ell(\Xi)-r)/2} s^{(\ell_0(\Gamma)-\ell_0(\Xi))/2} \mathrm{sig}(J(\Xi))$$

Note that this has an important consequence: the signature of $I(\Gamma)$ on each copy of $J(\Xi)$ contained in a given level r of the Jantzen filtration is the same.

Remark 8.15 Here is how I think of this. Assume the infinitesimal character is integral, so all orientation numbers are 0. The "default" level of the Jantzen filtration for $J(\Xi)$ to occur in is $\ell(\Gamma) - \ell(\Xi)$. If $J(\Xi)$ occurs in that level, it appears with + times its c-form. If it occurs instead in this shifted by $2k$, then the sign is $(-1)^k$. If the infinitesimal character isn't integral the same holds, except there is also an orientation number term.

This gives

$$w^c_{\Xi,\Gamma} = Q^c_{\Xi,\Gamma}(1) = s^{(\ell_0(\Gamma)-\ell_0(\Xi))/2} Q_{\Xi,\Gamma}(s). \tag{8.16}$$

which written out more explicitly is

$$w^c_{\Xi,\Gamma} = Q^c_{\Xi,\Gamma}(s, q = 1) = s^{(\ell_0(\Gamma)-\ell_0(\Xi))/2} Q_{\Xi,\Gamma}(q = s) \in \mathbb{Z}[s].$$

Lemma 8.17

$$\mathrm{sig}(I(\Gamma_{1+\epsilon})) = \mathrm{sig}(I(\Gamma_{1-\epsilon})) + (1-s) \sum_{r \text{ odd}} \mathrm{sig}(I(\Gamma), r)$$

See [1, Corollary 15.12(3)].

We now have the ingredients for the next result, cf. [1, Theorem 21.5(2)].

Proposition 8.18 *For small ϵ we have*

$$\mathrm{sig}(\Gamma_{1+\epsilon}) = \mathrm{sig}(\Gamma_{1-\epsilon}) + (1-s) \sum_{\substack{\Xi<\Gamma \\ \ell(\Gamma)-\ell(\Xi) \text{ odd}}} s^{(\ell_0(\Gamma)-\ell_0(\Xi))/2} Q_{\Xi,\Gamma}(s) \mathrm{sig}(J(\Xi))$$

Proof. This follows from (8.10) and (8.16). $\square$

We next want to express $\mathrm{sig}(J(\Xi))$ on the right hand side of the proposition in terms of signatures of standard modules. We first prove:

Lemma 8.19 *Define* $P^c_{\Xi,\Gamma} \in \mathbb{W}[q]$ *to be* $(-1)^{\ell(\Gamma)-\ell(\Xi)}$ *times the corresponding entry of the inverse of the* $Q^c_{\Xi,\Gamma}$ *matrix. Then*

$$P^c_{\Xi,\Gamma}(q) = s^{(\ell_0(\Gamma)-\ell_0(\Xi))/2} P_{\Xi,\Gamma}(sq)$$

See [1, Corollary 20.12].
Proof. The $P^c_{\Xi,\Gamma}$ are defined by the identity

$$\sum_{\Phi} (-1)^{\ell(\Xi)-\ell(\Phi)} P^c_{\Xi,\Phi} Q^c_{\Phi,\Gamma} = \delta_{\Xi,\Gamma}$$

Substitute (8.12):

$$\sum_{\Phi} (-1)^{\ell(\Xi)-\ell(\Phi)} s^{(\ell_0(\Gamma)-\ell_0(\Phi))/2} P^c_{\Xi,\Phi} Q_{\Phi,\Gamma}(sq) = \delta_{\Xi,\Gamma}$$

Substitute the proposed formula $P^c_{\Xi,\Phi} = s^{(\ell_0(\Phi)-\ell_0(\Xi))/2} P_{\Xi,\Phi}(sq)$ to give

$$\sum_{\Phi} (-1)^{\ell(\Xi)-\ell(\Phi)} s^{(\ell_0(\Gamma)-\ell_0(\Phi))/2} s^{(\ell_0(\Phi)-\ell_0(\Xi))/2} P_{\Xi,\Phi}(sq) Q_{\Phi,\Gamma}(sq) = \delta_{\Xi,\Gamma}$$

which simplifies to

$$s^{(\ell_0(\Gamma)-\ell_0(\Xi))/2} \sum_{\Phi} (-1)^{\ell(\Xi)-\ell(\Phi)} P_{\Xi,\Phi}(sq) Q_{\Phi,\Gamma}(sq) = \delta_{\Xi,\Gamma}$$

which is true, proving the Lemma. □

Lemma 8.20

$$\mathrm{sig}(J(\Gamma)) = \sum_{\Xi<\Gamma} s^{(\ell_0(\Gamma)-\ell_0(\Xi))/2} (-1)^{\ell(\Gamma)-\ell(\Xi)} P_{\Xi,\Gamma}(s) \mathrm{sig}(I(\Xi))$$

See [1, Corollary 20.13(1)].
Proof.
Since

$$\mathrm{sig}(\mathrm{gr}(I(\Gamma))) = \sum_{\Xi} Q^c_{\Xi,\Gamma}(v, q=1) \mathrm{sig}(J(\Xi))$$

by the definition of $P^c_{\Xi,\Gamma}$ this gives

$$\mathrm{sig}(J(\Gamma)) = \sum_{\Xi} (-1)^{\ell(\Gamma)-\ell(\Xi)} P^c_{\Xi,\Gamma}(1) \mathrm{sig}(I(\Xi))$$

and the result follows upon substituting (from Lemma 8.19):

$$P^c_{\Xi,\Gamma}(1) = s^{(\ell_0(\Gamma)-\ell_0(\Xi))/2} P_{\Xi,\Gamma}$$

$\square$

Theorem 8.21

$$\begin{aligned}\mathrm{sig}(\Gamma_{1+\epsilon}) = \mathrm{sig}(\Gamma_{1-\epsilon}) + \\ (1-s) \sum_{\substack{\Phi,\Xi \\ \Phi<\Xi<\Gamma \\ \ell(\Gamma)-\ell(\Xi) \text{ odd}}} s^{(\ell_0(\Gamma)-\ell_0(\Xi))/2} P_{\Phi,\Xi}(s) Q_{\Xi,\Gamma}(s)\mathrm{sig}(I(\Phi))\end{aligned}$$

Proof. By Proposition 8.18 and Lemma 8.20 $\mathrm{sig}(\Gamma_{1+\epsilon}) - \mathrm{sig}(\Gamma_{1-\epsilon})$ is equal to $(1-s)$ times:

$$\begin{aligned}
&\sum_{\substack{\Xi<\Gamma \\ \ell(\Gamma)-\ell(\Xi) \text{ odd}}} s^{(\ell_0(\Gamma)-\ell_0(\Xi))/2} Q_{\Xi,\Gamma}(s)\mathrm{sig}(J(\Xi)) = \\
&\sum_{\substack{\Xi<\Gamma \\ \ell(\Gamma)-\ell(\Xi) \text{ odd}}} s^{(\ell_0(\Gamma)-\ell_0(\Xi))/2} Q_{\Xi,\Gamma}(s) \sum_{\Phi<\Xi} s^{(\ell_0(\Xi)-\ell_0(\Phi))/2} P_{\Phi,\Xi}(s)\mathrm{sig}(I(\Phi)) \\
&= \sum_{\substack{\Phi,\Xi \\ \Phi<\Xi<\Gamma \\ \ell(\Gamma)-\ell(\Xi) \text{ odd}}} s^{(\ell_0(\Gamma)-\ell_0(\Xi))/2} s^{(\ell_0(\Xi)-\ell_0(\Phi))/2} P_{\Phi,\Xi}(s) Q_{\Xi,\Gamma}(s)\mathrm{sig}(I(\Phi)) \\
&= \sum_{\substack{\Phi,\Xi \\ \Phi<\Xi<\Gamma \\ \ell(\Gamma)-\ell(\Xi) \text{ odd}}} s^{(\ell_0(\Gamma)-\ell_0(\Phi))/2} P_{\Phi,\Xi}(s) Q_{\Xi,\Gamma}(s)\mathrm{sig}(I(\Phi))
\end{aligned}$$

$\square$

9 The Deformation Conjecture

Next we state the analogue of Lemma 8.17, which is due to Schmid and Vilonen. This is a guess about what their statement translates to here.

Conjecture 9.1

$$\mathrm{hodge}(I(\Gamma_{1+\epsilon})) = \mathrm{hodge}(I(\Gamma_{1-\epsilon})) + \sum_{r\ge 0}(1-v^r)\mathrm{hodge}(I(\Gamma),r)$$

Of all of the Conjectures we need this one appears to be on the firmest footing.

Remark 9.2 Keep in mind $\mathrm{hodge}(I(\Gamma),r)$ is the induced filtration on $\mathrm{grW}_r(I(\Gamma))$, which is computed at $t=1$, and the filtration is computed by taking a limit from $t>1$. Suppose we know $\mathrm{hodge}(I(\Gamma_{1-\epsilon}))$. We want to compute $\mathrm{hodge}(I(\Gamma_{1+\epsilon}))$, but to do this we need to know $\mathrm{hodge}(I(\Gamma_{1+\epsilon}))$ so that we can compute the other terms $\mathrm{hodge}(I(\Gamma),r)$. This is handled by the inductive algorithm, assuming Conjecture 6.8.

Note: the coefficient is $(1-v^r)$ instead of $(v^{-r}-1)$.

Proposition 9.3 *Assuming Conjecture 9.1 we have:*

$$\sum_{r\ge 0}(1-v^r)\mathrm{hodge}(I(\Gamma),r) = \sum_{\Xi\le\Gamma}\left[Q^H_{\Xi,\Gamma}(1) - v^{\ell(\Gamma)-\ell(\Xi)}Q^H_{\Xi,\Gamma}(v^{-2})\right]\mathrm{hodge}(J(\Xi))$$

To be clear: $Q^H_{\Xi,\Gamma}\in\mathbb{Z}[v,q]$, and on the right hand the terms appearing are $Q^H_{\Xi,\Gamma}(v,q=1)$ and $Q^H_{\Xi,\Gamma}(v,q=v^{-2})$, in $\mathbb{Z}[v]$.

Remark 9.4 Suppose the dimension of the orbit changes at a reducibility point. Because we've normalized our Hodge functions by multiplying by $v^{-a(\Gamma)}$ the deformation formula holds in this case without any further powers of v.

Remark 9.5 The left hand side of the identity in the Proposition, evaluated at a K-type μ, is a polynomial in v. We'll see shortly that $v^{\ell(\Gamma)-\ell(\Xi)}Q^H_{\Xi,\Gamma}(v^{-2})\in\mathbb{Z}[v]$ also.

Proof. Start by applying Definition 6.1.

$$\begin{aligned}
\sum_{r\geq 0}(1-v^r)\mathrm{hodge}(I(\Gamma),r) &= \sum_{r\geq 0}(1-v^r)\sum_{\Xi\leq\Gamma} w^{H,r}_{\Xi,\Gamma}J_v(\Xi)\\
&= \sum_{\Xi\leq\Gamma}\Big\{\sum_{r\geq 0}(1-v^r)w^{H,r}_{\Xi,\Gamma}\Big\}J_v(\Xi)\\
&= \sum_{\Xi\leq\Gamma}\Big\{\sum_{r\geq 0}(1-v^r)w^{H,r}_{\Xi,\Gamma}\Big\}J_v(\Xi)\\
&= \sum_{\Xi\leq\Gamma}\Big\{\sum_{r\geq 0}w^{H,r}_{\Xi,\Gamma}-\sum_{r\geq 0}v^r w^{H,r}_{\Xi,\Gamma}\Big\}J_v(\Xi)
\end{aligned}$$

Consider the terms inside the braces. Recall (6.3)

$$Q^H_{\Xi,\Gamma}(q) = \sum_{r\geq 0} w^{H,r}_{\Xi,\Gamma}q^{(\ell(\Gamma)-\ell(\Xi)-r)/2} \in \mathbb{Z}[v,q]$$

Therefore the first term in braces is

$$Q^H_{\Xi,\Gamma}(v,q=1).$$

For the second term compute

$$\begin{aligned}
Q^H_{\Xi,\Gamma}(v,q=v^{-2}) &= \sum_{r\geq 0} w^{H,r}_{\Xi,\Gamma}v^{(-\ell(\Gamma)+\ell(\Xi)+r)}\\
&= v^{-\ell(\Gamma)+\ell(\Xi)}\sum_{r\geq 0} w^{H,r}_{\Xi,\Gamma}v^r
\end{aligned} \tag{9.6}$$

i.e. the second term is

$$v^{\ell(\Gamma)-\ell(\Xi)}Q^H_{\Xi,\Gamma}(v,q=v^{-2}).$$

Plugging these in gives the result.

□

Let's do a consistency check. Consider Proposition 9.3, evaluated at $v=s$. This gives

$$(1-s)\sum_{r\text{ odd}}\mathrm{sig}(I(\Gamma),r) = (1-s)\sum_{\substack{\Xi\\ \ell(\Gamma)-\ell(\Xi)\text{ odd}}} Q^H_{\Xi,\Gamma}(v=s,q=1)\mathrm{sig}(\Xi)$$

Using using Conjecture 7.2(3) this gives

$$(1-s)\sum_{r \text{ odd}} \mathrm{sig}(I(\Gamma), r) = (1-s) \sum_{\substack{\Xi \\ \ell(\Gamma)-\ell(\Xi) \text{ odd}}} Q^c_{\Xi,\Gamma}\mathrm{sig}(\Xi)$$

which is precisely (8.10).

Proposition 9.7 *Assuming Conjectures 9.1 and 6.8 we have*

$$\begin{aligned} &\mathrm{hodge}(I(\Gamma_{1+\epsilon})) = \mathrm{hodge}(I(\Gamma_{1-\epsilon}))+ \\ &\quad \sum_{\Xi\le\Gamma} v^{(\ell_0(\Gamma)-\ell_0(\Xi<))/2}[Q_{\Xi,\Gamma}(v) - v^{\ell(\Gamma)-\ell(\Xi)}Q_{\Xi,\Gamma}(v^{-1})]J_v(\Xi)\end{aligned}$$

Proof. By conjecture 6.8

$$\begin{aligned} Q^H_{\Xi,\Gamma}(v, q=1) &= v^{(\ell_0(\Xi)-\ell_0(\Gamma)}Q_{\Xi,\Gamma}(v) \\ Q^H_{\Xi,\Gamma}(v, q=v^{-2}) &= v^{(\ell_0(\Xi)-\ell_0(\Gamma)}Q_{\Xi,\Gamma}(v^{-1})\end{aligned}$$

$\square$

Remark 9.8 Since $\deg(Q(v)) \le (\ell(\Gamma)-\ell(\Xi))/2$, $v^{-\ell(\Gamma)-\ell(\Xi)}Q(v^{-1}) \in \mathbb{Z}[v]$. I think $v^{(\ell_0(\Gamma)-\ell_0(\Xi))/2} \in \mathbb{Z}[v]$ but this requires proof.

Note that evaluating both sides at $v=s$ the Proposition reduces to (8.10) again.

Lemma 9.9 *Define* $P^H_{\Xi,\Gamma} \in \mathbb{Z}[v,q]$ *to be* $(-1)^{\ell(\Gamma)-\ell(\Xi)}$ *times the corresponding entry of the inverse of the* $Q^H_{\Xi,\Gamma}$ *matrix. Then*

$$P^H_{\Xi,\Gamma}(v,q) = v^{(\ell_0(\Gamma)-\ell_0(\Xi))/2}P_{\Xi,\Gamma}(vq)$$

Proof. The $P^H_{\Xi,\Gamma}$ are defined by the identity

$$\sum_{\Phi}(-1)^{\ell(\Xi)-\ell(\Phi)}P^H_{\Xi,\Phi}Q^H_{\Phi,\Gamma} = \delta_{\Xi,\Gamma}$$

Substitute (6.9)(b):

$$\sum_{\Phi}(-1)^{\ell(\Xi)-\ell(\Phi)}v^{(\ell_0(\Gamma)-\ell_0(\Phi))/2}P^H_{\Xi,\Phi}Q_{\Phi,\Gamma}(vq) = \delta_{\Xi,\Gamma}$$

Substitute the proposed formula $P^H_{\Xi,\Phi} = v^{(\ell_0(\Phi)-\ell_0(\Xi))/2}P_{\Xi,\Phi}(vq)$ to give

$$\sum_{\Phi}(-1)^{\ell(\Xi)-\ell(\Phi)}v^{(\ell_0(\Gamma)-\ell_0(\Phi))/2}v^{(\ell_0(\Phi)-\ell_0(\Xi))/2}P_{\Xi,\Phi}(vq)Q_{\Phi,\Gamma}(vq) = \delta_{\Xi,\Gamma}$$

which simplifies to

$$v^{(\ell_0(\Gamma)-\ell_0(\Xi))/2}\sum_{\Phi}(-1)^{\ell(\Xi)-\ell(\Phi)}P_{\Xi,\Phi}(vq)Q_{\Phi,\Gamma}(vq) = \delta_{\Xi,\Gamma}$$

which is true, proving the Lemma. □

Here is the analogue of [1, Corollary 20.13] which is also Lemma 8.19.

Proposition 9.10

$$\mathrm{hodge}(J(\Gamma)) = \sum_{\Xi\leq\Gamma}v^{(\ell_0(\Gamma)-\ell_0(\Xi))/2}(-1)^{\ell(\Gamma)-\ell(\Xi)}P_{\Xi,\Gamma}(v)\mathrm{hodge}(I(\Xi))$$

This is just like the proof of Lemma 8.19.

Proof. Since (Lemma 6.4(12)):

$$\mathrm{hodge}(I(\Gamma)) = \sum_{\Xi}Q^H_{\Xi,\Gamma}(v,q=1)\mathrm{hodge}(J(\Xi))$$

by the definition of $P^H_{\Xi,\Gamma}$ this gives

$$\mathrm{hodge}(J(\Gamma)) = \sum_{\Xi}(-1)^{\ell(\Gamma)-\ell(\Xi)}P^H_{\Xi,\Gamma}(v,q=1)\mathrm{hodge}(I(\Xi))$$

and insert $P^H_{\Xi,\Gamma}(v,q=1) = v^{(\ell_0(\Gamma)-\ell_0(\Xi))/2}P_{\Xi,\Gamma}(v)$ from Lemma 9.9. □

Next we substitue hodge$(J(\Xi))$ on the right of Proposition 9.7 with a sum over standard modules, and get an analogue of Lemma 8.20. (i.e. [1, Corollary 20.13(1)]).

Proposition 9.11 *Assuming Conjectures 9.1 and 6.8 we have*

$$\mathrm{hodge}(I(\Gamma_{1+\epsilon})) = \mathrm{hodge}(I(\Gamma_{1-\epsilon}))+$$
$$\sum_{\substack{\Phi,\Xi \\ \Phi\leq\Xi\leq\Gamma}} v^{(\ell_0(\Gamma)-\ell_0(\Phi))/2}(-1)^{\ell(\Xi)-\ell(\Phi)}P_{\Phi,\Xi}(v)[Q_{\Xi,\Gamma}(v) - v^{\ell(\Gamma)-\ell(\Xi)}Q_{\Xi,\Gamma}(v^{-1})]\mathrm{hodge}(I(\Phi))$$

This follows immediately upon plugging Proposition 9.10 into Proposition 9.7.

Here is a restatement.

Corollary 9.12 *Assuming Conjectures 9.1 and 6.8 we have*

$$\mathrm{hodge}(I(\Gamma_{1+\epsilon})) - \mathrm{hodge}(I(\Gamma_{1-\epsilon})) =$$
$$-\sum_{\Phi<\Gamma} v^{(\ell_0(\Gamma)-\ell_0(\Phi)/2}\left[\sum_{\substack{\Xi\\ \Phi\le\Xi\le\Gamma}} (-1)^{\ell(\Xi)-\ell(\Phi)} v^{\ell(\Gamma)-\ell(\Xi)} P_{\Phi,\Xi}(v) Q_{\Xi,\Gamma}(v^{-1})\right]\mathrm{hodge}(I(\Phi))$$

Proof. By the Proposition: $\mathrm{hodge}(I(\Gamma_{1+\epsilon})) - \mathrm{hodge}(I(\Gamma_{1-\epsilon}))$ is equal to

$$\sum_{\substack{\Phi,\Xi\\ \Phi\le\Xi\le\Gamma}} v^{(\ell_0(\Gamma)-\ell_0(\Phi)/2}(-1)^{\ell(\Xi)-\ell(\Phi)} P_{\Phi,\Xi}(v)[Q_{\Xi,\Gamma}(v) - v^{\ell(\Gamma)-\ell(\Xi)} Q_{\Xi,\Gamma}(v^{-1})]\mathrm{hodge}(I(\Phi))$$

$$= \sum_{\Phi\le\Gamma} v^{(\ell_0(\Gamma)-\ell_0(\Phi))/2}\left[\sum_{\Phi\le\Xi\le\Gamma} (-1)^{\ell(\Xi)-\ell(\Phi)} P_{\Phi,\Xi}(v) Q_{\Xi,\Gamma}(v)\right]\mathrm{hodge}(I(\Phi)) -$$
$$\sum_{\Phi\le\Gamma} v^{(\ell_0(\Gamma)-\ell_0(\Phi))/2}\left[\sum_{\Phi\le\Xi\le\Gamma} v^{\ell(\Gamma)-\ell(\Xi)}(-1)^{\ell(\Xi)-\ell(\Phi)} P_{\Phi,\Xi}(v) Q_{\Xi,\Gamma}(v^{-1})\right]\mathrm{hodge}(I(\Phi))$$

First of all

$$\sum_{\Phi\le\Xi\le\Gamma} P_{\Phi,\Xi}(v) Q_{\Xi,\Gamma}(v) = \delta_{\Phi,\Gamma} \quad \text{(Kronecker } \delta)$$

so the first sum over Φ is equal to $\mathrm{hodge}(I(\Gamma))$. The second sum over Φ is equal to

$$\mathrm{hodge}(I(\Gamma)) + \sum_{\Phi<\Gamma} v^{\ell(\Gamma)-\ell(\Phi)}[\dots]\mathrm{hodge}(I(\Phi)).$$

and the result follows. □

9.1 Summary

Here is a summary of an algorithm to compute $\mathrm{hodge}(X)$, for any irreducible or standard module, in terms of $\mathrm{hodge}(X_i)$ for X_i tempered. We continue to assume Conjectures 9.1 and 6.8.

Suppose X is a standard module. Write $X = I(\Gamma)$ with Γ final. By Corollary 9.12

$$\text{hodge}(I(\Gamma_{1+\epsilon})) = \text{hodge}(I(\Gamma_{1-\epsilon})) - \sum_{\Phi<\Gamma} v^{(\ell_0(\Gamma)-\ell_0(\Phi)/2}$$

(9.13)(a)
$$\Bigg[\sum_{\substack{\Xi \\ \Phi\le\Xi\le\Gamma}} (-1)^{\ell(\Xi)-\ell(\Phi)} v^{\ell(\Gamma)-\ell(\Xi)} P_{\Phi,\Xi}(v) Q_{\Xi,\Gamma}(v^{-1})\Bigg]\text{hodge}(I(\Phi))$$

we can compute $\text{hodge}(I(x,\lambda,\nu))$ in terms of terms $\text{hodge}(I(\Phi))$ with smaller ν. By induction this gives a formula for $\text{hodge}(I(\Gamma))$ in terms of $\text{hodge}(I(\Phi))$ for Φ tempered (i.e. $\nu = 0$).

Suppose π is an irreducible representation. Write $\pi = J(\Gamma)$ for Γ final. Then Proposition 9.10:

$$\text{hodge}(J(\Gamma)) = \sum_{\Xi\le\Gamma} v^{(\ell_0(\Gamma)-\ell_0(\Xi))/2}(-1)^{\ell(\Gamma)-\ell(\Xi)} P_{\Xi,\Gamma}(v)\text{hodge}(I(\Xi))$$

expresses $J(\Gamma)$ in terms of $\text{hodge}(I(\Xi))$, and the latter can be computed by the previous discussion in terms of tempered $I(\Phi)$.

Theorem 9.14 *Assume Conjectures 9.1 and 6.8. Suppose X is an irreducible or standard module with real infinitesimal character. There is a computable formula:*

$$\text{hodge}(X) = \sum_{j=1}^{n} w_j \text{hodge}(I_j)$$

where $w_j \in \mathbb{Z}[v]$ and each I_j is tempered with real infinitesimal character.

Furthermore write

$$\text{sig}(X) = \sum_{k=1}^{m} z_k \text{sig}(I'_k)$$

where $z_k \in \mathbb{Z}[s]$. Then $m = n$, the I_j and I'_k are the same, and

$$w_i(v = s) = z_i \quad (1 \le i \le n).$$

In other words

$$\text{hodge}(X)(v = s) = \text{sig}(X).$$

10 Hodge Filtrations of Tempered Representations

We now turn to the question of computing hodge($I(\Gamma)$) where $I(\Gamma)$ is an irreducible tempered representation.

The analogous question for the signature $\mathrm{sig}_h(I(\Gamma))$ of the Hermitian form is straightforward: $\mathrm{sig}_h(I(\Gamma))(\mu) = \mathrm{mult}(I(\Gamma))(\mu)$, and there is an effective algorithm to compute this. (Curious fact: the corresponding formula for the c-form is not so clear, and in fact isn't known in the unequal rank case. Never mind for now.)

Let $\mathcal{P}_t$ be the set of equivalence classes of non-zero, final, standard, tempered parameters with real infinitesimal character. Associated to $\Gamma \in \mathcal{P}_t$ is a non-zero, irreducible tempered representation $I(\Gamma)$ with real infinitesimal character Γ, with unique lowest K-type $\mu(\Gamma)$. We write $\mathcal{P}_t = \mathcal{P}_t(G)$ if we want to emphasize G.

The maps $\Gamma \to I(\Gamma)$ and $\mu(\Gamma)$ are bijections from $\mathcal{P}_t$ to the set of $I(\Gamma)'s$ and $\widehat{K}$, respectively. We refer to Γ or $I(\Gamma)$ as a "K-type". Sometimes we will refer to $\mu(\Gamma)$ as a "K-rep" to distinguish these notions.

Define $h(\Gamma,\tau) \in \mathbb{N}[v]$ for $\Gamma, \tau \in \mathcal{P}_t$ by:

$$\mathrm{hodge}(I(\Gamma)) = \sum_{\tau \in \mathcal{P}_t} h(\Gamma,\tau)\mu(\tau) \tag{10.1}$$

i.e.

$$\mathrm{hodge}(I(\Gamma)(\mu(\tau)) = h(\Gamma,\tau),$$

or equivalently

$$h(\Gamma,\tau) = \sum_{i=0}^{n} \mathrm{mult}(\mu(\tau), \mathrm{gr}_{i+a(\Gamma)}(I(\Gamma)))v^i$$

where $a(\Gamma)$ is the codimension of the underlying orbit. Our main objective is to compute $h(\Gamma,\tau)$.

The matrix $h(\Gamma,\tau)$ of polynomials is upper unitriangular, and therefore invertible over $\mathbb{Z}[v]$. Define $\{H(\Gamma,\tau) \in \mathbb{Z}[v]\}$ to be the inverse matrix. That is:

$$\mathrm{hodge}(\mu(\tau)) = \sum_{\Gamma} H(\Gamma,\tau)\mathrm{hodge}(I(\tau)) \tag{10.2}$$

(this is a *finite* sum). The left hand side is the irreducible representation $\mu(\tau)$ with trivial Hodge filtration. This means

$$\mathrm{hodge}(\mu)(\mu') = \begin{cases} 1 & \mu = \mu' \\ 0 & \text{otherwise} \end{cases}$$

(as opposed to $\mathrm{hodge}(\mu)(\mu) = v^a$). Note that (10.2) evaluated at $v = 1$ gives the *K-type formula*

$$\mu = \sum_{i=1}^{n} a_i I(\Gamma_i)|_K$$

which is realized by the `atlas` command `K_type_formula`.

Our strategy to compute $h(\Gamma, \tau)$ is to compute the polynomials $H(\Gamma, \tau)$. Then the $h(\Gamma, \tau)$ are obtained by taking the inverse. However the induction is going to involve passing back and forth between then two.

11 Bott-Borel-Weil induction

Suppose $Q = LU$ is a θ-stable parabolic of G and σ is a finite dimensional (algebraic) representation of $L \cap K$. This defines a vector bundle $\mathcal{S}_\sigma$ on $K/Q \cap K$.

Definition 11.1 *Define*

$$\text{BBW-Ind}_{L\cap K}^{K}(\sigma) = \sum_i (-1)^i H^i(K/Q \cap K, \mathcal{S}_\sigma)$$

This is virtual K-module.

If K is connected and σ is irreducible then $\text{BBW-Ind}_{L\cap K}^{K}(\sigma)$ is irreducible or 0. In general all we know is that all of the constituents of $\text{BBW-Ind}_{L\cap K}^{K}(\sigma)$ have the same infinitesimal character. This shouldn't cause any trouble.

Definition 11.2 *Suppose π is an $L \cap K$-module, equipped with a grading gr, with grading function $f : \widehat{K} \to \mathbb{Z}[v]$. Then BBW-Ind($\pi$) has a natural grading function BBW-Ind(f). If we write*

$$\mathrm{gr}(\pi)|_{L\cap K} = \sum_{\mu_L \in \widehat{L\cap K}} f(\mu_L)\mu_L$$

then
$$\mathrm{gr}(\text{BBW-Ind}_{L\cap K}^{K}(\pi))|_K = \sum_{\mu_L\in\widehat{L\cap K}} f(\mu_L)\,\text{BBW-Ind}_{L\cap K}^{K}(\mu_L)$$

An equivalent statement is

$$\text{BBW-Ind}_{L\cap K}^{K}(f)(\mu) = \sum_{\mu_L} f(\mu_L)\mathrm{mult}(\mu,\text{BBW-Ind}_{L\cap K}^{K}(\mu_L))$$

If K is connected the sum on the right has only one term, and this can be written
$$\text{BBW-Ind}_{L\cap K}^{K}(f)(\mu) = f(\mu_L)$$
where $\text{BBW-Ind}_{L\cap K}^{K}(\mu_L) = \mu$.

12 Cohomological Induction and Standardization

Suppose $Q = LU$ is a θ-stable parabolic and Γ_L is a parameter for L. We write Ind_Q^G for cohomological induction.

Write $\mathfrak{q} = l \oplus \mathfrak{u}$. The version of cohomological induction we're using takes infinitesimal character γ to infinitesimal character $\gamma + \rho_G - \rho_L = \gamma + \rho(\mathfrak{u})$. Write
$$\Gamma_L = (x_L, \lambda_L, \nu_L).$$
This has infinitesimal character $\gamma_L = \frac{1+\theta_{x_L}}{2}\lambda_L + \nu_L$ (assuming $\theta_{x_L}(\nu_L) = \nu_L$). Let's define cohomological induction of parameters as follows.

Definition 12.1

(12.2)(a) $$\mathrm{Ind}_L^G(\Gamma_L) = \Gamma$$

where

(12.2)(b) $$\Gamma = (x_G, \lambda_G, \nu_L)$$

where

$$\texttt{x\_G} = \texttt{embed\_KGB}(\texttt{x\_L})$$

and

$$\lambda = \lambda_L + \rho_G - \rho_L$$

Then Γ is a parameter for G with infinitesimal character

$$\gamma_G = \frac{1+\theta_{x_G}}{2}(\lambda_L + \rho(G) - \rho(L)) + \nu_L = \gamma_L + \rho_G - \rho_L$$

and is standard if

$$\langle \alpha^\vee, \gamma_L + \rho_G - \rho_L \rangle \geq 0 \quad \alpha^\vee \in \Delta^\vee(\mathfrak{u}).$$

Then $I(\Gamma)$ is defined, this is a standard module if Γ is standard, and otherwise is a continued standard module, defined by coherent continuation.

Furthermore

$$\mathrm{Ind}_Q^G(\Gamma_L) = I_G(\Gamma).$$

If this isn't standard then `standardize` replaces it with a `ParamPol` of standard modules, in fact `theta_induce_standard` has this built in.

In our application we're going to have $\nu_L = 0$. In any event we need a formula

$$\mathrm{hodge}(I_G(\Gamma)) = \sum_i w_i \mathrm{hodge}_G(I_G(\Gamma_i))$$

where $w_i \in \mathbb{Z}[v]$ and $\Gamma_i \in \mathcal{P}_t$. If Γ is standard with $\nu = 0$ (i.e. $\Gamma \in \mathcal{P}_t$) then there's nothing to do. In general we apply `hodge_normalize` to $I_G(\Gamma)$. We recall briefly what this does. This is described in more detail in the Appendix.

If $I(\Gamma)$ isn't normal there may be some simple complex roots α, which are descents (i.e. of type `C-`) which are singular on the infinitesimal character of Γ. We simply replace Γ with $s_\alpha(\Gamma)$, without introducing any powers of v. See the discussion before `hodge_reflection_complex` in hodge_normalize.at. This is implemented in `hodge_reflection_complex` in hodge_normalization.at.

If λ is not dominant with respect to some non-compact imaginary root then we use a graded Hecht-Schmid identity. This is described in the pdf file above, and implemented in `hodge_reflection_imaginary` in hodge_normalize.at

Conclusion: If $Q = LU$ is θ-stable and $\Gamma_L \in \mathcal{P}_t(L)$ then there is an algorithm to write

$$\mathrm{hodge}(\mathrm{Ind}_Q^G(\Gamma_L)) = \sum_i w_i \mathrm{hodge}(I_G(\Gamma_i))$$

where $w_i \in \mathbb{Z}[v]$ and $\Gamma_i \in \mathcal{P}_t$.

13 Further Results of Schmid and Vilonen

If G is split (by which we mean the derived group is split) let Γ^0 be the parameter of the spherical principal series $I_G(\Gamma^0)$ of G with infinitesimal character 0.

First of all Schmid and Vilonen tell us the Hodge filtration on $I_G(\Gamma^0)$. Let $\mathcal{N}$ be the nilpotent cone in $\mathfrak{g}$, and $\mathcal{N}_\theta = \mathcal{N} \cap \mathfrak{s}$. Write $\mathcal{R}(\mathcal{N}_\theta)$ for the algebraic functions on this complex algebraic variety.

Proposition 13.1 (Schmid-Vilonen)

$$\mathrm{gr}(I_G(\Gamma^0)) \simeq \mathrm{gr}(\mathcal{R}(\mathcal{N}_\theta))$$

The left hand side is the associated graded of the Hodge filtration, and the right hand side is the natural grading of $\mathcal{R}(\mathcal{N}_\theta)$.

Suppose $\Gamma \in \mathcal{P}_t$. Then there is a θ-stable parabolic $Q = LU$, with L split, and a one-dimensional representation μ_L of L, such that the following holds. Then

$$\mu(\Gamma) = \text{BBW-Ind}_{L\cap K}^{K}(\mu_L) \tag{13.2}$$

and

$$I(\Gamma) = \mathrm{Ind}_Q^G(I_L(\mu_L)) \tag{13.3}$$

The pair (Q, μ_L) is given by the `atlas` command `tau_q@K_Type`. Perhaps it is helpful to write

$$I_L(\mu_L) = I_L(\Gamma_L^0) \otimes \mu_L$$

On the right hand side $\Gamma_L^0 = (x_{\max}, \vec{0}, \vec{0})$, and

$$\mu_L = \Gamma_L^0 \otimes \mu_L = (x_{\max}, \frac{1+\theta_x}{2}\mu_L, \frac{1-\theta_x}{2}\mu_L)$$

Lemma 13.4

$$\mathrm{Ind}_Q^G(I_L(\mu_L))|_K \simeq \text{BBW-Ind}_{L\cap K}^{K}(I_L(\mu_L) \otimes \mathcal{S}(\mathfrak{u} \cap \mathfrak{s})).$$

This is a restatement of Zuckerman's version of the Blattner formula.

Remark 13.5 This lemma is written more narrowly than is necessary. It is a statement about cohomological induction, and I think it holds as long as the cohomology vanishes except in the middle degree.

Also the one-dimensional μ_L has the trivial grading. Then define the natural grading on $I_L(\mu_L) \otimes \mathcal{S}(\mathfrak{u} \cap \mathfrak{s})$ to be

$$\text{hodge}(I_L(\mu_L) \otimes \text{symm}(\mathfrak{u} \cap \mathfrak{s}))$$

(see (3.1)). The k^{th} level is

$$\sum_{p+q=k} \text{gr}_p(I_L(\mu_L) \otimes \mathcal{S}^q(\mathfrak{u} \cap \mathfrak{s}).$$

The next input we need is

Proposition 13.6 (Schmid-Vilonen) *Assume* $\Gamma \in \mathcal{P}_t$. *Let* (Q, μ_L) *be defined as above. Recall* $Q = LU$, L *is split,* μ_L *is a one-dimensional representation of* L, *and* $I(\Gamma) = Ind_Q^G(I_L(\mu_L))$. Then

$$\text{BBW-Ind}_{L\cap K}^K(\text{hodge}(I_L(\mu_L)) \otimes \text{symm}(\mathfrak{u} \cap \mathfrak{s})) = \text{hodge}(I(\Gamma))$$

The left hand side is the Hodge grading function of $I(\Gamma)$. The right hand side is the function just discussed, induced up to $\widehat{K}$ by Definition 11.2.

It is helpful (though perhaps not essential) to have the same conclusion with $I_L(\mu_L)$ replaced by a more general standard module on L.

Suppose $\Gamma_L = (x_L, \lambda, \vec{0}) \in \mathcal{P}_t(L)$. Let $\Gamma = \text{Ind}_L^G(\Gamma_L)$ (Definition 12.1) Recall Γ is standard if

$$\langle \alpha^\vee, \gamma_L + \rho_G - \rho_L \rangle \geq 0 \quad \alpha^\vee \in \Delta^\vee(\mathfrak{u}) \tag{13.7}$$

Corollary 13.8 *Assume* (13.7) *holds. Then*

$$\text{BBW-Ind}_{L\cap K}^K(\text{hodge}(I_L(\Gamma_L)) \otimes \text{symm}(\mathfrak{u} \cap \mathfrak{s})) = \text{hodge}(I(\Gamma)).$$

This is just Proposition 13.6 with a more general standard module on L. It comes down to induction by stages.

Proof. We can write

$$I_L(\Gamma_L) = \mathrm{Ind}_{Q_L}^{L}(I_M(\Gamma_M^0) \otimes \mu_M) \tag{13.9}$$

where $Q_L = MU_L$ is a θ-stable parabolic in L, M is split, Γ_M^0 is spherical, and μ_M is a one-dimensional representation of M. By induction by stages

$$I(\Gamma) = \mathrm{Ind}_{Q_G}^{G}(I_M(\Gamma_M^0) \otimes \mu_M)$$

where $Q_G = MU_G$ is a θ-stable parabolic in G, with $\mathfrak{u}_G = \mathfrak{u} \oplus \mathfrak{u}_L$. By Proposition 13.6 we have

(13.10)(a)

$$\mathrm{hodge}(I(\Gamma) = \text{BBW-Ind}_{M\cap K}^{K}(\mathrm{hodge}(I_M(\Gamma_M^0)) \otimes \mu_M \otimes \mathrm{symm}(\mathfrak{u}_G \cap \mathfrak{s})))$$

We want to show this equals

(13.10)(b) $$\text{BBW-Ind}_{L\cap K}^{K}(\mathrm{hodge}(I_L(\Gamma_L)) \otimes \mathrm{symm}(\mathfrak{u} \cap \mathfrak{s}))$$

By Proposition 13.6 applied now to L, using (13.9), we have

$$\mathrm{hodge}(I_L(\Gamma_L)) = \text{BBW-Ind}_{M\cap K}^{L\cap K}(\mathrm{hodge}(I_M(\Gamma_M^0) \otimes \mu_M \otimes \mathrm{symm}(\mathfrak{u}_L \cap \mathfrak{s})))$$

Plugging this in to (a), and using $\mathrm{symm}(\mathfrak{u}_L\cap\mathfrak{s})\otimes\mathrm{symm}(\mathfrak{u}\cap\mathfrak{s}) = \mathrm{symm}(\mathfrak{u}_G\cap\mathfrak{s})$ we recover (a). □

We *hope* we can drop the assumption that $I(\Gamma)$ is standard as follows.

Conjecture 13.11

$$\text{BBW-Ind}_{L\cap K}^{K}(\mathrm{hodge}(I_L(\Gamma_L))\otimes\mathrm{symm}(\mathfrak{u}\cap\mathfrak{s})) = \texttt{hodge\_normalize}(\mathrm{hodge}(I(\Gamma)))$$

This is perhaps the Conjecture for which we have the least evidence and is the primary obstacle to proving our algorithm is complete. It involves computing the Hodge filtration on a module constructed from a $\mathcal{D}_\lambda$ module where λ is not dominant. We handle this by doing wall crossing; the formulas we use are given in the Appendix, which are based on calculations in rank 1.

Note that the conjecture implies that if Γ is standard then

$$\text{BBW-Ind}_{L\cap K}^{K}(\mathrm{hodge}(I_L(\Gamma_L)) \otimes \mathrm{symm}(\mathfrak{u} \cap \mathfrak{s})) = \mathrm{hodge}(I(\Gamma))$$

which is slightly stronger than Corollary 13.8.

14 Some more formalism

Before stating the algorithm it is helpful to make introduce some more formalism, and state things in `atlas` terms.

Here are some data types in `atlas`.

(1) a `hodgeParamPol` is a sum $P = \sum_{i=1}^{n} a_i\Gamma_i$ where $a_i \in \mathbb{Z}[v]$ and $\Gamma_i \in \mathcal{P}_t$. It represents the Hodge function

$$\sum_{i=1}^{n} a_i \mathrm{hodge}(I(\Gamma_i))$$

i.e. the Hodge function

$$\mathcal{P}_t \ni \Xi \mapsto \sum_{i=1}^{n} a_i \mathrm{hodge}(\mu(\Xi), I(\Gamma_i)) \in \mathbb{Z}[v].$$

Every Hodge function of interest is represented by a `hodgeParamPol`.

Side note: the actual data type in atlas is `hodgeParamPol=[ParamPol]`. See `hodgeParamPol.at`.

(2) a `KHodgeParamPol` is also essentially a sum $KP = \sum_{i=1}^{n} a_i\Gamma_i$; to distinguish this from a `hodgeParamPol` we append a `void`: a `KHodgeParamPol` is a pair `(hodgeParamPol,void)`. This represents a finite sum of K-rep's with $\mathbb{Z}[v]$ coefficients:

$$KP = \sum_{i=1}^{n} a_i\Gamma_i \longleftrightarrow \sum_{i=1}^{n} a_i\mu(\Gamma_i).$$

(3) a `hodgeFunction` is a function, usually written f, from $\mathcal{P}_t \simeq \widehat{K}$ to $\mathbb{Z}[v]$.

We are primarily interested in Hodge functions. However these are infinite objects, so we need to understand how to work with them in terms of hodgeParamPols and KHParamPols.

Suppose S is a finite subset of $\mathcal{P}_t$.

Consider the following diagram:

$$\begin{tikzcd}
\{\text{hodgeParamPols}\} \arrow[rr, "\Omega"] \arrow[dr, "\Phi_S"'] & & \{\text{Hodge functions}\} \arrow[dl, "\Psi_S"] \\
& \{\text{KHodgeParamPols}\} &
\end{tikzcd}$$

The upper arrow is the definition of `hodgeParamPol` in (1). That is each hodgeParamPol represents a Hodge function:

$$\Omega(\sum_{i=1}^{n} a_i\Gamma_i) = \sum_{i=1}^{n} a_i \text{hodge}(I(\Gamma_i)).$$

The Hodge functions we're interested in are all in the image of this map.

The other two maps depend on S as indicated. The map from from Hodge functions to KHodgeParamPols is:

$$\Psi_S(f) = \sum_{\Xi\in S} f(\Xi)\Xi \longleftrightarrow \sum_{\Xi\in S} f(\Xi)\mu(\Xi)$$

The map Φ_s is defined to make the diagram commute:

$$\Phi_S(\sum_{i=1}^{n} a_i\Gamma_i) = \sum_{\Xi\in S}\sum_{i=1}^{n} a_i \text{hodge}(\mu(\Xi), I(\Gamma_i))\mu(\Xi)$$

With this setup we can state one of the issues that arises. Suppose P is a hodgeParamPol and f is a Hodge function. How big does S need to be to conclude

$$\Phi_S(P) = \Psi_S(f) \Rightarrow \Omega(P) = f?$$

A related question is the following. Suppose f is a Hodge function. How do we find a hodgeParamPol P so that $\Omega(P) = f$? What we do is: given S, we find P so that

$$\Phi_S(P) = \Psi_S(f)$$

and then argue that S is large enough to imply $\Omega(P) = f$.

15 The Algorithm I

Our goal is to compute $H(\Gamma, \tau)$ defined by (the finite sum):

$$\text{hodge}(\mu(\tau)) = \sum_{\Gamma} H(\Gamma, \tau)\text{hodge}(I(\tau)).$$

So fix $\Gamma \in \mathcal{P}_t$ and let $\mu = \mu(\Gamma) \in \widehat{K}$. Recall (13.2) we're given $Q = LU$, with L split, and a one-dimensional representation μ_L of L such that

$$\mu = \text{BBW-Ind}_{L\cap K}^{L}(\mu_L)$$

Write the trivial representation of L by the Zuckerman character formula

$$\mathbb{C}_L = \sum_j a_j I_L(\phi_j)$$

where each $I_L(\phi_j)$ is a standard module for L (not necessarily with $\nu = 0$). Tensor with μ_L:

$$\mu_L = \sum_j a_j I_L(\phi_j) \otimes \mu_L \tag{15.1)(a}$$

or

$$\mu_L = \sum_j a_j I_L(\psi_j) \tag{15.1)(b}$$

where $\psi_j = \phi_j \otimes \mu_L$. This is given by the `atlas` command `character_formula_one_dimensional` (which does not require the KLV polynomials). After applying the Hodge deformation algorithm `hodge_recursive_deform` to the right hand side we obtain a formula

$$\mathrm{hodge}(\mu_L) = \sum_j m_j \mathrm{hodge}(I_L(\Gamma_{L,j})) \tag{15.2}$$

where $\Gamma_{L,j} \in \mathcal{P}_t(L)$, and $m_j \in \mathbb{Z}[v]$. We believe this formula is correct in all cases.

Now we introduce the graded Koszul identity to give:

$$\mathrm{hodge}(\mu_L) = \sum_j m_j \mathrm{hodge}(I_L(\Gamma_{L,j})) \otimes \mathrm{symm}(\mathfrak{u} \cap \mathfrak{s}) \otimes \mathrm{alt}(\mathfrak{u} \cap \mathfrak{s}) \tag{15.3}$$

Now compute $\text{BBW-Ind}_{L\cap K}^K$ of both sides (Definition 11.2). The left hand side is simply $\mathrm{hodge}(\mu)$, so

$$\begin{aligned}\mathrm{hodge}(\mu) = \sum_j m_j \, \text{BBW-Ind}_{L\cap K}^K \big\{ & \mathrm{hodge}(I_L(\Gamma_{L,j})) \otimes \\ & \mathrm{symm}(\mathfrak{u} \cap \mathfrak{s}) \otimes \mathrm{alt}(\mathfrak{u} \cap \mathfrak{s}) \big\}\end{aligned} \tag{15.4}$$

Let's look at a summand of the right hand side:

$$\text{BBW-Ind}_{L\cap K}^K(\mathrm{hodge}(I_L(\Gamma_{L,j})) \otimes \mathrm{symm}(\mathfrak{u} \cap \mathfrak{s}) \otimes \mathrm{alt}(\mathfrak{u} \cap \mathfrak{s}))$$

We apply Proposition 13.6 to $\mathrm{hodge}(I_L(\Gamma_{L,j}))$. There is a θ-stable parabolic $Q_j = L_jU_j$ of L, with Lie algebra $\mathfrak{q}_j = l_j \oplus \mathfrak{u}_j$, and a one-dimensional representation μ_{L_j} of L_j, such that L_j is split, and with $I_{L_j}(\Gamma^0_{L,j})$ the spherical representation of L_j with infinitesimal character 0, we have (see (13.3)):

$$I_L(\Gamma_{L,j}) = \mathrm{Ind}_{Q_j}^L(I_{L_j}(\mu_{L_j}))$$

Then Proposition 13.6 gives

$$\mathrm{hodge}(I_L(\Gamma_{L,j})) = \text{BBW-Ind}_{L_j\cap K}^{L\cap K}(\mathrm{hodge}(I_{L_j}(\mu_L)) \otimes \mathrm{symm}(\mathfrak{u}_j \cap \mathfrak{s}))$$

Plugging these into (15.4) we conclude

$$\begin{aligned}\mathrm{hodge}(\mu) = &\sum_j m_j\, \text{BBW-Ind}_{L\cap K}^{K}(\text{BBW-Ind}_{L_j\cap K}^{L\cap K}\big\{\\ &\mathrm{hodge}(I_{L_j}(\mu_{L_j})) \otimes \mathrm{symm}(\mathfrak{u}_j \cap \mathfrak{s}) \otimes \mathrm{symm}(\mathfrak{u} \cap \mathfrak{s}) \otimes \mathrm{alt}(\mathfrak{u} \cap \mathfrak{s}))\big\}\end{aligned}$$

Now let $\mathfrak{u}_j^G = \mathfrak{u}_j \oplus \mathfrak{u}$, and $\mathfrak{q}_j^G = l_j \oplus \mathfrak{u}_j^G$. Then induction by stages for BBW-Ind gives (using $\mathrm{symm}(V \oplus W) \simeq \mathrm{symm}(V) \otimes \mathrm{symm}(W)$):

$$\mathrm{hodge}(\mu) = \sum_j m_j\, \text{BBW-Ind}_{L_j\cap K}^{K}\, \mathrm{hodge}(I_{L_j}(\mu_{L_j}))\otimes\mathrm{symm}(\mathfrak{u}_j^G\cap\mathfrak{s})\otimes\mathrm{alt}(\mathfrak{u}\cap\mathfrak{s})))$$

We're going to need to know how to compute

$$\text{BBW-Ind}_{L_j\cap K}^{K}(\mathrm{hodge}(I_{L_j}(\mu_{L_j}))) \otimes \mathrm{symm}(\mathfrak{u}_j^G \cap \mathfrak{s}) \otimes \mathrm{alt}(\mathfrak{u} \cap \mathfrak{s}))$$

We're going to proceed by induction on the group. So let's *assume* that for each j we can find a formula

$$\mathrm{hodge}(I_{L_j}(\mu_{L_j})) \otimes \mathrm{alt}(\mathfrak{u} \cap \mathfrak{s})) = \sum_i w_i \mathrm{hodge}(I_{L_j}(\Gamma_{j,i})) \tag{15.5}$$

with $w_i \in \mathbb{Z}[v]$ and $\Gamma_{j,i} \in \mathcal{P}_t(L_j)$. We'll return to this step in Section 17. Also it does not arise for complex groups, so in this case we are done (Section 16). We reiterate this equation is on L_j, which is split, and μ_{L_j} is one-dimensional.

Remark 15.6 Recall $L_j \subset L \subset G$ and $\mathfrak{u}_j^G = \mathfrak{u}_j \oplus \mathfrak{u}$, so $\mathfrak{u}$ (appearing on the left hand side of (15.5)) is not the full nilpotent radical of the parabolic $l_j \oplus \mathfrak{u}_j^G \subset \mathfrak{g}$.

We conclude

$$\text{(15.7)}\quad \mathrm{hodge}(\mu) = \sum_j \sum_i m_j w_i \,\text{BBW-Ind}(\mathrm{hodge}(I_{L_j}(\Gamma_{j,i}) \otimes \mathrm{symm}(\mathfrak{u}_j^G \cap \mathfrak{s}))$$

Then by Corollary 13.11 we have

$$\mathrm{hodge}(\mu) = \sum_j m_j \sum_i w_i \texttt{hodge\_normalize}(\mathrm{hodge}(\mathrm{Ind}_{L_j}^G(I_{L_j}(\Gamma_{j,i}))))$$

16 Case of Complex groups

Suppose $G(\mathbb{R})$ is a complex group. Then the only split Levi factor is the Cartan subgroup $H \simeq \mathbb{C}^{\times n}$, with corresponding parabolic a Borel $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{u}$. Suppose $\Gamma \in \mathcal{P}_t$ and $\mu = \mu(\Gamma)$. Then $\mu = \text{BBW-Ind}_{H\cap K}^K(\mu_L)$ where μ_L is a character of H. Then (15.4) says

$$\mathrm{hodge}(\mu) = \text{BBW-Ind}_{H\cap K}^K(\mu_L \otimes \mathrm{symm}(\mathfrak{u}\cap\mathfrak{s}) \otimes \mathrm{alt}(\mathfrak{u}\cap\mathfrak{s}))$$

So the inductive step in this case is simply writing

$$\mathrm{hodge}(\mu_L) \otimes \mathrm{alt}(\mathfrak{u}\cap\mathfrak{s}) = \sum_k (-1)^k \sum_{\tau \in \bigwedge^k(\mathfrak{u}\cap\mathfrak{s})} \mathrm{hodge}(\mu_L \otimes \tau)$$

This takes care of the inductive step so the algorithm is complete.

17 The inductive step

Let's return to the inductive step (15.5). Here is the situation, from section 15, with some notation changed.

We're given split Levi factors L_J, L:

$$L_j \subset L \subset G$$

and a θ-stable parabolic

$$l \oplus \mathfrak{u} \subset \mathfrak{g}.$$

Then $\bigwedge^\bullet(\mathfrak{u}\cap\mathfrak{s})$ is a graded representation of $L \cap K$, and by restriction $L_j \cap K$. We're given a one-dimensional representation μ_{L_j} of L_j.

We need a formula

$$\text{(17.1)} \qquad \mathrm{hodge}(I_{L_j}(\mu_{L_j})) \otimes \mathrm{alt}(\mathfrak{u} \cap \mathfrak{s}) = \sum a_i \mathrm{hodge}(I_{L_j}(\Gamma_{L_j,i}))$$

where $a_i \in \mathbb{Z}[v]$ and $\Gamma_{L_j,i} \in \mathcal{P}_t(L_j)$. We do this calculation for each k, that is we need to compute

$$\text{(17.2)} \qquad \mathrm{hodge}(I_{L_j}(\mu_{L_j})) \otimes \mathrm{alt}^k(\mathfrak{u} \cap \mathfrak{s}) = \sum_{i=0}^{n} a_i \mathrm{hodge}(I_{L_j}(\Gamma_{L_j,i}))$$

with $a_i \in \mathbb{Z}[v]$, where alt^k has a factor of $(-1)^k$, and occurs in degree k. By induction on the group we may assume we know $\mathrm{hodge}(I_{L_j}(\mu_{L_j}))$ as a function on $\widehat{L \cap K}$. One point is we need to be careful: in the software we'll be handed a Hodge function, but we don't want to evaluate it a K-type except when absolutely necessary. Let's order the terms on the right hand side by height, so $I_{L_j}(\Gamma_{L_j,0}))$ is the smallest term.

First of all we need a crucial special case (which will get us out of our inductive loop).

Lemma 17.3 *Suppose $L_1 = G$. Then $\mathfrak{u}_1 = 0$ and the formula of* (17.2) *is simply*

$$\mathrm{hodge}(I(\Gamma_L)) = 1 * \mathrm{hodge}(I_L(\Gamma_L))$$

This is precisely what is what is needed for the algorithm to proceed.

We have $L_j \subset L \subset G$, and we're trying to find a formula

$$\text{(17.4)} \qquad \mathrm{hodge}(I_{L_j}(\mu_{L_j})) \otimes \mathrm{alt}^k(\mathfrak{u} \cap \mathfrak{s}) = \sum_{i=0}^{n} a_i \mathrm{hodge}(I_{L_j}(\Gamma_{L_j,i})).$$

I've changed notation slightly: we're fixing k, and writing a_i in place of $a_{k,i}$.

If $L = G$ there is nothing to do (see Corollary 13.11), so assume $L \subsetneq G$.

We've fixed an element $\gamma^\vee$ to define the height of K-types for G and each Levi subgroup. If G is semisimple we can take $\gamma^\vee = \rho^\vee(G)$. Call this function $\mathrm{ht}_{\gamma^\vee}$.

Now we fix an integer N. We need to make sure N is big enough. More on this later.

We start with the set S_L of $L \cap K$-types as follows. Compute

$$\text{(17.5)} \qquad I_{L_j}(\mu_{L_j}) \otimes \bigwedge^k(\mathfrak{u} \cap \mathfrak{s}) = \sum_r a_r I(\Gamma_{L_j,r})$$

where $\Gamma_{L_j,r} \in \mathcal{P}_t(L_j)$. This is a straightforward $L_j \cap K$-type calculation (nothing involving Hodge filtrations). In atlas this is, with P the parabolic and p the parameter on L_j:

```
set weights=sums_nci_nilrad_roots_wedge_k_restricted_to_H_theta(P,k)
add_weights(p,weights)
```

Then let S_0 be the set of all $L_j \cap K$-types occuring in this formula, out to height N. That is run over all standard modules occuring on the right hand side of (17.5), and for each one compute all of the $L_j \cap K$ types τ with $\mathrm{ht}_{\gamma^\vee}(\tau) \le N$.

Inductively, we're going to be given a triple (f, S) consisting of a Hodge function for L_j, a set of $L_j \cap K$-types, and a HodgeParamPol P. We start with

$$(\mathrm{hodge}(I_L(\mu_L)) \otimes \mathrm{alt}^k(\mathfrak{u}_1 \cap \mathfrak{s}), S_0, 0).$$

For the inductive step let τ be an $L_j \cap K$-type of minimal $\mathrm{ht}_{\gamma^\vee}$ in S, and let $w = f(\tau) \in \mathbb{Z}[v]$. Then we replace

$$(f, S, P) \longrightarrow (f - w * \mathrm{hodge}(I_{L_j}(\tau)), S', P + w * I_{L_j}(\tau))$$

where S' is obtained from S by adding all $L_j \cap K$-types of $I_{L_j}(\tau)$ up to $\mathrm{ht}_{\gamma^\vee} \le N$.

The algorithm concludes when $f(\tau) = 0$ for all $\tau \in S$.

We need to make sure N is large enough so that when the algorithm concludes we have

$$\Omega(P) = f.$$

A first guess is

$$N = \max_{\phi} \langle \gamma^\vee, \phi \rangle$$

where we run over all weights ϕ of $\bigwedge^k(\mathfrak{u} \cap \mathfrak{s})$. See Section 14.

We conclude with a conjecture which should follow from the preceding discussion.

Conjecture 17.6 *Suppose π is an irreducible tempered representation. Then*

$$hodge(\pi)(v = s) = sig(\pi) \tag{17.7}$$

Assuming this, Theorem 9.14 implies (17.7) *relation holds for all irreducible representations.*

18 Appendix: Hodge Normalization

18.1 Normal parameters

Suppose (x, λ, ν) is a parameter. We only are interested in the case $\nu = 0$, so we assume this from now on.

Remark 18.1.1 Even though $\nu = 0$, the relevant notion of *normal* is in the setting of `repr` (parameters for representations of G), not that of `standardrepk` (K-types).

Recall the infinitesimal character is

$$\gamma = \frac{(1 + \theta_x)\lambda}{2}$$

The parameter p is *normal* if γ is (weakly) dominant, and there are no singular complex descents. That is $\alpha > 0$ implies

(18.1.2)(a) $$\langle \gamma, \alpha \rangle \geq 0$$

and if α is θ_x-complex then

(18.1.2)(b) $$\langle \gamma, \alpha^\vee \rangle = 0 \text{ implies } \theta_x(\alpha) > 0.$$

18.2 Complex roots

Assume α is θ_x-complex and

$$\langle \gamma, \alpha^\vee \rangle \leq 0$$

Let $\beta = \theta_x(\alpha)$. Note that (since $\nu = 0$)

$$\langle \gamma, \alpha^\vee \rangle = \langle \gamma, \beta^\vee \rangle.$$

Lemma 18.2.3 *Suppose α is a simple root, $\langle \gamma, \alpha^\vee \rangle \leq 0$, and α is a complex descent. That is*

$$\begin{aligned} \theta_x(\alpha) = \beta &\leq 0 \\ \langle \gamma, \alpha^\vee \rangle &\leq 0 \\ \langle \gamma, -\beta^\vee \rangle &\geq 0 \end{aligned}$$

Then

$$\text{hodge}(I(x, \lambda, 0)) = \text{hodge}(I(s_\alpha x, s_\alpha \lambda, 0))$$

Remark 18.2.4 This is the claim at least if α, β are in the special complex root system (orthogonal to $\rho_i^\vee, \rho_r^\vee$), in particular $\langle \alpha, \beta^\vee \rangle = 0$. I'm not sure if this should hold more generally.

Remark 18.2.5 Using the Lemma we can assume there are no singular complex descents.

Assume $\langle \gamma, \alpha^\vee \rangle < 0$ and
$$\langle \alpha, \beta^\vee \rangle = 0.$$
Then after the change we are in the following situation. Write $\gamma' = s_\alpha \gamma$, $x' = s_\alpha x$. Then:
$$\begin{aligned} \theta_{x'}(\alpha) &= -\beta > 0 \\ \langle \gamma', \alpha^\vee \rangle &> 0 \\ \langle \gamma', -\beta^\vee \rangle &> 0 \end{aligned}$$
so now the parameter is normal with respect to $\{\alpha, -\beta\}$ (and α is a $\theta_{x'}$-ascent).

If $\langle \alpha, \beta^\vee \rangle \neq 0$ it is more complicated.

Lemma 18.2.6 *Suppose α is a simple roots, $\langle \gamma, \alpha^\vee \rangle < 0$, and α is a complex ascent. That is*
$$\begin{aligned} \theta_x(\alpha) &= \beta > 0 \\ \langle \gamma, \alpha^\vee \rangle &< 0 \\ \langle \gamma, \beta^\vee \rangle &< 0 \end{aligned}$$
Let
$$n = -2\langle \gamma, \alpha^\vee \rangle \in \mathbb{Z}_{>0}$$
Then

(18.2.7)(a)
$$\begin{aligned} \mathrm{hodge}(I(x, \lambda, 0)) = v\,\mathrm{hodge}(I_H(s_\alpha x, s_\alpha \lambda, 0)) + \\ \sum_{k=1}^{[n/2]-1} v^{k-1}(v^2 - 1)\mathrm{hodge}(I(s_\alpha x, s_\alpha \lambda - k\alpha, 0)) + \\ [(v-1)v^{n/2}\mathrm{hodge}(I(s_\alpha x, s_\alpha \lambda - \frac{n}{2}\alpha, 0))] \end{aligned}$$

where the last term occurs if and only if n is even.

Suppose $\langle\alpha,\beta^\vee\rangle = 0$. Then after the change we are in the following situation. Write $\gamma' = s_\alpha\gamma$, $x' = s_\alpha x$. Then:

$$\begin{aligned}\theta_{x'}(\alpha) &= -\beta < 0\\ \langle\gamma',\alpha^\vee\rangle &> 0\\ \langle\gamma',\beta^\vee\rangle &< 0\end{aligned}$$

Switching the roles of α,β we write this

$$\begin{aligned}\theta_{x'}(\beta) &= -\alpha < 0\\ \langle\gamma',\beta^\vee\rangle &< 0\\ \langle\gamma',\alpha^\vee\rangle &> 0\end{aligned}$$

and we're back in the setting of Lemma 18.2.3. So Lemmas 18.2.6 and 18.2.3 imply:

Lemma 18.2.8 *Suppose $\langle\gamma,\alpha^\vee\rangle < 0$ and α is a complex ascent. That is*

$$\begin{aligned}\theta_x(\alpha) &= \beta > 0\\ \langle\gamma,\alpha^\vee\rangle &< 0\\ \langle\gamma,\beta^\vee\rangle &< 0\end{aligned}$$

Furthermore assume

$$\langle\alpha,\beta^\vee\rangle = 0.$$

Let $w = s_\alpha s_\beta$. Then:

$$\begin{aligned}\text{hodge}(I(x,\lambda,0)) = v\text{hodge}(I_H(x,w\lambda,0))+&\\ \sum_{k=1}^{[n/2]-1} v^{k-1}(v^2-1)\text{hodge}(I(x,w\lambda-k\alpha,0))+&\\ [(v-1)v^{n/2}\text{hodge}(I(x,w\lambda-\frac{n}{2}\alpha,0))]&\end{aligned} \tag{18.2.9}$$

Note that x has not changed.

Note that after applying w we have,

$$\begin{aligned}\theta_x(\alpha) &= \beta > 0\\ \langle w\gamma,\beta^\vee\rangle &> 0\\ \langle w\gamma,\alpha^\vee\rangle &> 0\end{aligned}$$

18.3 Imaginary roots

Suppose α is a simple, noncompact imaginary root, and

$$\langle\gamma,\alpha^\vee\rangle < 0$$

Lemma 18.3.10 *Suppose α is a simple root, $\langle\gamma,\alpha^\vee\rangle \le 0$, and α is noncompact, imaginary.*

Recall there is a single Cayley transform $c^\alpha x$.

Suppose α is type I, i.e. $s_\alpha(x) \neq x$. The Cayley transform of the parameter $I(x,\lambda,0)$ is single valued:

$$c^\alpha I(x,\lambda,0) = I(c^\alpha x,\lambda,0)$$

If n is even then

$$\begin{aligned}\mathrm{hodge}(I(x,\lambda,0)) =& v^{\frac{n}{2}}\mathrm{hodge}(I(c^\alpha x,\lambda,0)) - \\ & vI(s_\alpha x, s_\alpha\lambda,0) - \\ & \sum_{k=1}^{\frac{n}{2}-1} v^{k-1}(v^2-1)I(s_\alpha x,\lambda+(n-k)\alpha,0) - \\ & v^{\frac{n}{2}-1}(v-1)I(s_\alpha x,\lambda+\frac{n}{2}\alpha)\end{aligned}$$

If n is odd then

$$\begin{aligned}\mathrm{hodge}(I(x,\lambda,0)) =& v^{[\frac{n}{2}]}\mathrm{hodge}(I(c^\alpha x,\lambda,0)) - \\ & vI(s_\alpha x, s_\alpha\lambda,0) - \\ & \sum_{k=1}^{[\frac{n}{2}]} v^{k-1}(v^2-1)I(s_\alpha x,\lambda+(n-k)\alpha,0)\end{aligned}$$

Now suppose α is type II, i.e. $s_\alpha x = x$. In this case the Cayley transform on the level of parameters is double-valued. The two values are $I(c^\alpha x,\lambda,0)$ and $s_\alpha \times I(c^\alpha x,\lambda,0) = I(c^\alpha x,\lambda+\alpha,0)$.

If n is even then

$$\begin{aligned}\text{hodge}(I(x,\lambda,0)) =& v^{[\frac{n}{2}]}\text{hodge}(I(c^\alpha x,\lambda,0)) + v^{[\frac{n}{2}]}\text{hodge}(I(c^\alpha x,\lambda+\alpha,0)) - \\ & vI(x,s_\alpha\lambda,0) - \\ & \sum_{k=1}^{\frac{n}{2}-1} v^{k-1}(v^2-1)\text{hodge}(I(x,\lambda+(n-k)\alpha,0)) - \\ & v^{\frac{n}{2}-1}(v-1)\text{hodge}(I(x,\lambda+\frac{n}{2}\alpha))\end{aligned}$$

If n is odd then

$$\text{hodge}(I(x,\lambda,0)) = v^{[\frac{n}{2}]}\text{hodge}(I(c^\alpha x,\lambda,0) + v^{[\frac{n}{2}]}\text{hodge}(I(c^\alpha x,\lambda+\alpha,0) -$$

$$\begin{aligned}& vI(x,s_\alpha\lambda,0) + \\ & \sum_{k=1}^{[\frac{n}{2}]} v^{k-1}(v^2-1)\text{hodge}(I(x,\lambda+(n-k)\alpha,0))\end{aligned}$$

These can be written uniformly as follows.

Lemma 18.3.11 *Suppose α is a simple root, $\langle\gamma,\alpha^\vee\rangle \leq 0$, and α is noncompact imaginary. Let $c^\alpha(I(x,\lambda,0))$ be the sum of the Cayley transforms of $I(x,\lambda,0)$:*

$$c^\alpha(I(x,\lambda,0)) = \begin{cases} I(c^\alpha x,\lambda,0) & \alpha \text{ type } I \\ I(c^\alpha x,\lambda,0) + I(c^\alpha x,\lambda+\alpha,0) & \alpha \text{ type } II. \end{cases}$$

Define

$$\tau(n,k) = \begin{cases} 1 & k = n/2 \\ 2 & \text{otherwise} \end{cases}$$

Then

$$\begin{aligned}\text{hodge}(I(x,\lambda,0)) =& v^{[\frac{n}{2}]}\text{hodge}(c^\alpha(I(x,\lambda,0)) - \\ & vI(s_\alpha x,s_\alpha\lambda,0) - \\ & \sum_{k=1}^{[\frac{n}{2}]} v^{k-1}(v^{\tau(n,k)}-1)I(s_\alpha x,\lambda+(n-k)\alpha,0)\end{aligned}$$

19 Examples

Here is a brief introduction to the terminology in the examples. Further details will be given in each example.

We work with a given connected complex group G equipped with a Cartan involution θ, and $K = G^\theta$ (a complex, possibly disconnected group). We also fix a fundamental Cartan subgroup H and Borel subgroup $B \supset H$. If G is not equal rank we are also given a distinguished involution δ, which is the Cartan involution of the quasi-compact form of G.

A parameter in `atlas` is a triple `(x,lambda,nu)`, where:

(1) `x` is a `KGB` element, i.e. a K-conjugacy class of Borel subgroups of G. Associated to x is an involution θ_x of H.

(2) `lambda` is in $X^*(H)+\rho$, and defines a character of (the ρ-cover of) H^{θ_x}.

(3) `nu` is in $(X^*(H)\otimes\mathbb{Q})^{-\theta_x}$, and defines a (real) character of $(H^{-\theta_x})^0$.

Associated to a parameter `p=(x,lambda,nu)` is a standard module `I(p)`, with a unique irreducible quotient `J(p)`. The infinitesimal character is

$$\gamma = \frac{1}{2}(1+\theta_x)\lambda + \nu$$

These modules have real infinitesimal character, and are tempered if and only if $\nu = 0$. There is a notion of equivalence of parameters.

The equivalence classes of parameters `(x,lambda,0)` parametrize irreducible tempered representations with real infinitesimal character. By taking the lowest K-type these also parametrize $\widehat{K}$. Thus by a *K-type* we mean a pair (`x`, `lambda`).

This parametrization of $\widehat{K}$ takes disconnectedness of K into account. If K is connected it is helpful to think instead in terms of highest weights. Even if K is not connected it is useful to consider highest weights of K^0.

The tables will specify three things about a K-type: `x,lambda` and `hwt`, the last being a highest weight for K^0. We try, not always succesfully, to choose a convenient set of positive roots for K. We also give the dimension of the K-type.

We write the Hodge function on a module as a formal infinite sum $\sum_\mu a(\mu)\mu$ with $a(\mu)\in\mathbb{Z}[v]$. We display the terms of such a formula up to a given height of the K-types.

Here is a typical table giving the Hodge function on a representation, in this case the spherical tempered representation of $SL(2,\mathbb{R})$:

```
atlas> show_long (hodge_branch_std(p,10))
c      codim  x  lambda    hwt       dim  height  mu
+1     0      2  [ 1 ]/1   [ 0 ]     1    0       0
+v     1      1  [ 1 ]/1   [ -2 ]    1    1       -1
+v     1      0  [ 1 ]/1   [ 2 ]     1    1       1
+v^2   1      1  [ 3 ]/1   [ -4 ]    1    3       -2
```

Here is an explanation of the columns:

`c` : coefficient, in $\mathbb{Z}[v]$

`codim` : codimension of K-orbit on G/B

`x` : `KGB` element, i.e. K-orbit on G/B

`lambda` : `(x,lambda)` is a K-type

`hwt` : highest weight of K-type (*)

`dim` : dimension of K-type (not K^0-type)

`height` : height of the K-type

`mu` : element of $\mathbb{Z}/2\mathbb{Z}$ needed to convert from c-form to Hermitian form

(*) The highest weight is an element of $X^*(H)$ whose restriction to $(H^\theta)^0$ is the highest weight of an irreducible representation of K^0.

19.1 $SL(2,\mathbb{R})$

Here is the Hodge filtration on the irreudicble, spherical, tempered representation of $SL(2,\mathbb{R})$.

```
atlas> set G=SL(2,R)
Variable G: RealForm
atlas> set p=trivial(G)*0
Variable p: Param
atlas> p
```

```
Value: final parameter(x=2,lambda=[1]/1,nu=[0]/1)
atlas> infinitesimal_character (p*0)
Value: [ 0 ]/1
atlas> show_long (hodge_branch_std(p,10))
c      codim  x  lambda   hwt       dim  height  mu
+1     0      2  [ 1 ]/1  [ 0 ]     1    0       0
+v     1      1  [ 1 ]/1  [ -2 ]    1    1       -1
+v     1      0  [ 1 ]/1  [ 2 ]     1    1       1
+v^2   1      1  [ 3 ]/1  [ -4 ]    1    3       -2
+v^2   1      0  [ 3 ]/1  [ 4 ]     1    3       2
+v^3   1      1  [ 5 ]/1  [ -6 ]    1    5       -3
+v^3   1      0  [ 5 ]/1  [ 6 ]     1    5       3
+v^4   1      1  [ 7 ]/1  [ -8 ]    1    7       -4
+v^4   1      0  [ 7 ]/1  [ 8 ]     1    7       4
+v^5   1      1  [ 9 ]/1  [ -10 ]   1    9       -5
+v^5   1      0  [ 9 ]/1  [ 10 ]    1    9       5
```

In other words the Hodge function, if we write $[k]$ for the representation $e^{ik\theta}$ of K, is:

$$\ldots v^3[-6] + v^2[-4] + v[-2] + [0] + v[2] + v^2[4] + v^3[6] \ldots$$

The Hodge filtration doesn't change for $0 \leq \nu < 1$. There is a jump at $\nu = 1$:

```
atlas> set p=trivial(G)
Variable p: Param
atlas> p
Value: final parameter(x=2,lambda=[1]/1,nu=[1]/1)
atlas> infinitesimal_character
Error during analysis of expression at <standard input>:15:0-23
  Undefined identifier 'infinitesimal_character'
Expression analysis failed
atlas> set p=trivial(G)
Variable p: Param
atlas> p
Value: final parameter(x=2,lambda=[1]/1,nu=[1]/1)
atlas> infinitesimal_character(p)
Value: [ 1 ]/1
```

```
atlas> show_long (hodge_branch_std(p,10))
c      codim  x  lambda   hwt      dim  height  mu
+1     0      2  [ 1 ]/1  [ 0 ]    1    0       0
+1     1      1  [ 1 ]/1  [ -2 ]   1    1       -1
+1     1      0  [ 1 ]/1  [ 2 ]    1    1       1
+v     1      1  [ 3 ]/1  [ -4 ]   1    3       -2
+v     1      0  [ 3 ]/1  [ 4 ]    1    3       2
+v^2   1      1  [ 5 ]/1  [ -6 ]   1    5       -3
+v^2   1      0  [ 5 ]/1  [ 6 ]    1    5       3
+v^3   1      1  [ 7 ]/1  [ -8 ]   1    7       -4
+v^3   1      0  [ 7 ]/1  [ 8 ]    1    7       4
+v^4   1      1  [ 9 ]/1  [ -10 ]  1    9       -5
+v^4   1      0  [ 9 ]/1  [ 10 ]   1    9       5
```

That is:

$$\dots v^2[-6] + v[-4] + [-2] + [0] + [2] + v[4] + v^2[6] \dots$$

Here is the finite dimensional representation of G with dimension 11, of course its Hodge polynomial is identically 1.

```
atlas> set p=finite_dimensional(G,[10])
Variable p: Param
atlas> dimension(p)
Value: 11
atlas> p
Value: final parameter(x=2,lambda=[1]/1,nu=[11]/1)
atlas> print_branch_irr(p,20)
atlas> show_long (hodge_branch_irr (p,12))
c   codim  x  lambda   hwt     dim  height  mu
+1  0      2  [ 1 ]/1  [ 0 ]   1    0       0
+1  1      1  [ 1 ]/1  [ -2 ]  1    1       -1
+1  1      0  [ 1 ]/1  [ 2 ]   1    1       1
+1  1      1  [ 3 ]/1  [ -4 ]  1    3       -2
+1  1      0  [ 3 ]/1  [ 4 ]   1    3       2
+1  1      1  [ 5 ]/1  [ -6 ]  1    5       -3
+1  1      0  [ 5 ]/1  [ 6 ]   1    5       3
+1  1      1  [ 7 ]/1  [ -8 ]  1    7       -4
+1  1      0  [ 7 ]/1  [ 8 ]   1    7       4
```

```
+1  1       1  [ 9 ]/1  [ -10 ]  1    9        -5
+1  1       0  [ 9 ]/1  [ 10 ]   1    9        5
```

19.2 $GL(3,\mathbb{R})$

This applies to $SL(3,\mathbb{R})$ well. Even though $GL(3,\mathbb{R})$ is disconnected it is more convenient because the coordinates are more natural (and the disconnectedness is inessential because $G(\mathbb{R}) = G(\mathbb{R})^0 Z(G(\mathbb{R}))$).

Here is the Hodge filtration on the irreducible tempered spherical representation:

```
atlas> set G=GL(3,R)
Variable G: RealForm
atlas> rho(G)
Value: [  1,  0, -1 ]/1
atlas> set p=trivial(G)*0
Variable p: Param
atlas> p
Value: final parameter(x=3,lambda=[1,0,-1]/1,nu=[0,0,0]/1)
atlas> infinitesimal_character (p)
Value: [ 0, 0, 0 ]/1
atlas> show_long (hodge_branch_std(p,20))
c                          codim x  lambda             hwt              dim  height mu
+1                         0     3  [  1,  0, -1 ]/1  [ 0, 0, 0 ]      1    0      0
+v+v^2                     2     0  [ 1, 1, 0 ]/1     [  1,  2, -1 ]  5    2      3
+v^3                       2     0  [  1,  0, -1 ]/1  [  1,  1, -2 ]  7    4      4
+v^2+v^3+v^4               2     0  [  2,  1, -1 ]/1  [  2,  2, -2 ]  9    6      5
+v^4+v^5                   2     0  [  2,  0, -2 ]/1  [  2,  1, -3 ]  11   8      6
+v^3+v^4+v^5+v^6           2     0  [  3,  1, -2 ]/1  [  3,  2, -3 ]  13   10     7
+v^5+v^6+v^7               2     0  [  3,  0, -3 ]/1  [  3,  1, -4 ]  15   12     8
+v^4+v^5+v^6+v^7+v^8       2     0  [  4,  1, -3 ]/1  [  4,  2, -4 ]  17   14     9
+v^6+v^7+v^8+v^9           2     0  [  4,  0, -4 ]/1  [  4,  1, -5 ]  19   16     10
+v^5+v^6+v^7+v^8+v^9+v^10 2     0  [  5,  1, -4 ]/1  [  5,  2, -5 ]  21   18     11
+v^7+v^8+v^9+v^10+v^11     2     0  [  5,  0, -5 ]/1  [  5,  1, -6 ]  23   20     12
```

The next 5 tables show the Hodge filtration on the spherical representation of $SL(3,\mathbb{R})$, with infinitesimal character $\nu = t\rho$, at the first 5 reducibility points. We list these as $\nu = s*(3\rho)$ where $s = 1/6, 1/3, 1/2, 5/6, 1$, i.e. $t = 1/2, 1, 3/2, 5/2, 3$.

The main point is that at each reducibility point the level of each K-type in the Hodge filtration potentially goes down; in particular the 0^{th} level gets bigger. In fact in the limit as $\nu \to \infty$ every K-type is in the 0^{th} level of the Hodge filtration.

```
atlas> set p=trivial(G)*3
Variable p: Param
atlas> infinitesimal_character (p)
Value: [  3,  0, -3 ]/1
atlas> set r=reducibility_points (p)
Variable r: [rat]
atlas> r
Value: [1/6,1/3,1/2,5/6,1/1]
atlas> void:for c in r do prints(new_line,q*c);show_long (hodge_branch_std(q*c,20)) od

final parameter(x=3,lambda=[1,0,-1]/1,nu=[1,0,-1]/2)
c                      codim  x  lambda            hwt               dim  height  mu
+1                     0      3  [  1,  0, -1 ]/1  [ 0, 0, 0 ]       1    0       0
+2v                    2      0  [ 1, 1, 0 ]/1     [  1,  2, -1 ]  5    2       3
+v^2                   2      0  [  1,  0, -1 ]/1  [  1,  1, -2 ]  7    4       4
+2v^2+v^3              2      0  [  2,  1, -1 ]/1  [  2,  2, -2 ]  9    6       5
+v^3+v^4               2      0  [  2,  0, -2 ]/1  [  2,  1, -3 ]  11   8       6
+2v^3+v^4+v^5          2      0  [  3,  1, -2 ]/1  [  3,  2, -3 ]  13   10      7
+v^4+v^5+v^6           2      0  [  3,  0, -3 ]/1  [  3,  1, -4 ]  15   12      8
+2v^4+v^5+v^6+v^7      2      0  [  4,  1, -3 ]/1  [  4,  2, -4 ]  17   14      9
+v^5+v^6+v^7+v^8       2      0  [  4,  0, -4 ]/1  [  4,  1, -5 ]  19   16      10
+2v^5+v^6+v^7+v^8+v^9  2      0  [  5,  1, -4 ]/1  [  5,  2, -5 ]  21   18      11
+v^6+v^7+v^8+v^9+v^10  2      0  [  5,  0, -5 ]/1  [  5,  1, -6 ]  23   20      12
```

```
final parameter(x=3,lambda=[1,0,-1]/1,nu=[1,0,-1]/1)
c                      codim  x  lambda              hwt               dim  height  mu
+1                     0      3  [  1,  0, -1 ]/1  [ 0, 0, 0 ]       1    0       0
+2                     2      0  [ 1, 1, 0 ]/1     [  1,  2, -1 ]  5    2       3
+1                     2      0  [  1,  0, -1 ]/1  [  1,  1, -2 ]  7    4       4
+3v                    2      0  [  2,  1, -1 ]/1  [  2,  2, -2 ]  9    6       5
+v+v^2                 2      0  [  2,  0, -2 ]/1  [  2,  1, -3 ]  11   8       6
+3v^2+v^3              2      0  [  3,  1, -2 ]/1  [  3,  2, -3 ]  13   10      7
+v^2+v^3+v^4           2      0  [  3,  0, -3 ]/1  [  3,  1, -4 ]  15   12      8
+3v^3+v^4+v^5          2      0  [  4,  1, -3 ]/1  [  4,  2, -4 ]  17   14      9
+v^3+v^4+v^5+v^6       2      0  [  4,  0, -4 ]/1  [  4,  1, -5 ]  19   16      10
+3v^4+v^5+v^6+v^7      2      0  [  5,  1, -4 ]/1  [  5,  2, -5 ]  21   18      11
+v^4+v^5+v^6+v^7+v^8  2      0  [  5,  0, -5 ]/1  [  5,  1, -6 ]  23   20      12
```

```
final parameter(x=3,lambda=[1,0,-1]/1,nu=[3,0,-3]/2)
c                   codim  x  lambda              hwt               dim  height  mu
+1                  0      3  [  1,  0, -1 ]/1  [ 0, 0, 0 ]       1    0       0
+2                  2      0  [ 1, 1, 0 ]/1     [  1,  2, -1 ]  5    2       3
+1                  2      0  [  1,  0, -1 ]/1  [  1,  1, -2 ]  7    4       4
+1+2v               2      0  [  2,  1, -1 ]/1  [  2,  2, -2 ]  9    6       5
+2v                 2      0  [  2,  0, -2 ]/1  [  2,  1, -3 ]  11   8       6
+v+3v^2             2      0  [  3,  1, -2 ]/1  [  3,  2, -3 ]  13   10      7
+2v^2+v^3           2      0  [  3,  0, -3 ]/1  [  3,  1, -4 ]  15   12      8
+v^2+3v^3+v^4       2      0  [  4,  1, -3 ]/1  [  4,  2, -4 ]  17   14      9
+2v^3+v^4+v^5       2      0  [  4,  0, -4 ]/1  [  4,  1, -5 ]  19   16      10
+v^3+3v^4+v^5+v^6  2      0  [  5,  1, -4 ]/1  [  5,  2, -5 ]  21   18      11
+2v^4+v^5+v^6+v^7  2      0  [  5,  0, -5 ]/1  [  5,  1, -6 ]  23   20      12
```

```
final parameter(x=3,lambda=[1,0,-1]/1,nu=[5,0,-5]/2)
c                codim  x  lambda            hwt              dim  height  mu
+1               0      3  [  1,  0, -1 ]/1  [ 0, 0, 0 ]      1    0       0
+2               2      0  [ 1, 1, 0 ]/1     [  1,  2, -1 ]  5    2       3
+1               2      0  [  1,  0, -1 ]/1  [  1,  1, -2 ]  7    4       4
+1+2v            2      0  [  2,  1, -1 ]/1  [  2,  2, -2 ]  9    6       5
+2v              2      0  [  2,  0, -2 ]/1  [  2,  1, -3 ]  11   8       6
+2v+2v^2         2      0  [  3,  1, -2 ]/1  [  3,  2, -3 ]  13   10      7
+3v^2            2      0  [  3,  0, -3 ]/1  [  3,  1, -4 ]  15   12      8
+2v^2+3v^3       2      0  [  4,  1, -3 ]/1  [  4,  2, -4 ]  17   14      9
+3v^3+v^4        2      0  [  4,  0, -4 ]/1  [  4,  1, -5 ]  19   16      10
+2v^3+3v^4+v^5   2      0  [  5,  1, -4 ]/1  [  5,  2, -5 ]  21   18      11
+3v^4+v^5+v^6    2      0  [  5,  0, -5 ]/1  [  5,  1, -6 ]  23   20      12
```

```
final parameter(x=3,lambda=[1,0,-1]/1,nu=[3,0,-3]/1)
c               codim  x  lambda            hwt              dim  height  mu
+1              0      3  [  1,  0, -1 ]/1  [ 0, 0, 0 ]      1    0       0
+2              2      0  [ 1, 1, 0 ]/1     [  1,  2, -1 ]  5    2       3
+1              2      0  [  1,  0, -1 ]/1  [  1,  1, -2 ]  7    4       4
+3              2      0  [  2,  1, -1 ]/1  [  2,  2, -2 ]  9    6       5
+2              2      0  [  2,  0, -2 ]/1  [  2,  1, -3 ]  11   8       6
+2+2v           2      0  [  3,  1, -2 ]/1  [  3,  2, -3 ]  13   10      7
+1+2v           2      0  [  3,  0, -3 ]/1  [  3,  1, -4 ]  15   12      8
+3v+2v^2        2      0  [  4,  1, -3 ]/1  [  4,  2, -4 ]  17   14      9
+v+3v^2         2      0  [  4,  0, -4 ]/1  [  4,  1, -5 ]  19   16      10
+3v^2+3v^3      2      0  [  5,  1, -4 ]/1  [  5,  2, -5 ]  21   18      11
+v^2+3v^3+v^4   2      0  [  5,  0, -5 ]/1  [  5,  1, -6 ]  23   20      12
```

19.3 $Sp(4,\mathbb{R})$

Here are the spherical representations of $Sp(4,\mathbb{R})$ with $\nu = 0$ (tempered) and $\nu = \frac{1}{2}, \frac{1}{3}, 1$ (the reducibility points).

```
atlas> set G=Sp(4,R)
Variable G: RealForm
atlas> set p=trivial(G)*0
Variable p: Param
atlas> p
Value: final parameter(x=10,lambda=[2,1]/1,nu=[0,0]/1)
atlas> set x=KGB(G,2)
Variable x: KGBElt
atlas> show_long (hodge_branch_std(p,10))
c              codim  x   lambda      hwt          dim  height  mu
+1             0      10  [ 2, 1 ]/1  [ 0, 0 ]     1    0       0
+v^2           2      4   [ 1, 0 ]/1  [ 1, 1 ]     3    2       4
+v^2           3      3   [ 1, 0 ]/1  [ -2,  2 ]   1    3       2
+v^2           3      2   [ 1, 0 ]/1  [  2, -2 ]   1    3       4
+v+v^3         3      1   [ 1, 0 ]/1  [ 0, 2 ]     3    3       1
+v+v^3         3      0   [ 1, 0 ]/1  [ 2, 0 ]     3    3       5
+v^4           2      6   [ 2, 1 ]/1  [ -1,  3 ]   3    6       2
+v^4           2      5   [ 2, 1 ]/1  [  3, -1 ]   3    6       6
+v^2+2v^4      2      4   [ 2, 1 ]/1  [ 2, 2 ]     5    6       6
+v^3+v^5       3      1   [ 2, 1 ]/1  [ 1, 3 ]     5    7       1
+v^3+v^5       3      0   [ 2, 1 ]/1  [ 3, 1 ]     5    7       7
+v^3+v^5       3      3   [ 3, 0 ]/1  [ -2,  4 ]   3    9       3
+v^3+v^5       3      2   [ 3, 0 ]/1  [  4, -2 ]   3    9       7
+v^2+v^4+v^6   3      1   [ 3, 0 ]/1  [ 0, 4 ]     5    9       2
+v^2+v^4+v^6   3      0   [ 3, 0 ]/1  [ 4, 0 ]     5    9       8
+v^4+2v^6      2      4   [ 3, 2 ]/1  [ 3, 3 ]     7    10      8
```

```
atlas> set p=trivial(G)
Variable p: Param
atlas> reducibility_points (p)
Value: [1/3,1/2,1/1]
atlas> show_long (hodge_branch_std(p*(1/3),10),x)
c             codim  x   lambda      hwt          dim  height  mu
+1            0      10  [ 2, 1 ]/1  [ 0, 0 ]     1    0       0
+v            2      4   [ 1, 0 ]/1  [  1, -1 ]   3    2       4
+v^2          3      3   [ 1, 0 ]/1  [ -2, -2 ]   1    3       2
+v^2          3      2   [ 1, 0 ]/1  [ 2, 2 ]     1    3       4
+v+v^2        3      1   [ 1, 0 ]/1  [  0, -2 ]   3    3       1
+v+v^2        3      0   [ 1, 0 ]/1  [ 2, 0 ]     3    3       5
+v^3          2      6   [ 2, 1 ]/1  [ -1, -3 ]   3    6       2
+v^3          2      5   [ 2, 1 ]/1  [ 3, 1 ]     3    6       6
+v^2+2v^3     2      4   [ 2, 1 ]/1  [  2, -2 ]   5    6       6
+v^2+v^4      3      1   [ 2, 1 ]/1  [  1, -3 ]   5    7       1
+v^2+v^4      3      0   [ 2, 1 ]/1  [  3, -1 ]   5    7       7
+v^3+v^4      3      3   [ 3, 0 ]/1  [ -2, -4 ]   3    9       3
+v^3+v^4      3      2   [ 3, 0 ]/1  [ 4, 2 ]     3    9       7
+v^2+v^3+v^5  3      1   [ 3, 0 ]/1  [  0, -4 ]   5    9       2
+v^2+v^3+v^5  3      0   [ 3, 0 ]/1  [ 4, 0 ]     5    9       8
+v^3+2v^5     2      4   [ 3, 2 ]/1  [  3, -3 ]   7    10      8
```

```
atlas> set q=trivial(G)*(1/2)
Variable q: Param
atlas> q
Value: final parameter(x=10,lambda=[2,1]/1,nu=[2,1]/2)
atlas> show_long (hodge_branch_std(q,10),x)
c           codim  x   lambda      hwt          dim  height  mu
+1          0      10  [ 2, 1 ]/1  [ 0, 0 ]     1    0       0
+v          2      4   [ 1, 0 ]/1  [  1, -1 ]  3    2       4
+v          3      3   [ 1, 0 ]/1  [ -2, -2 ]  1    3       2
+v          3      2   [ 1, 0 ]/1  [ 2, 2 ]     1    3       4
+2v         3      1   [ 1, 0 ]/1  [  0, -2 ]  3    3       1
+2v         3      0   [ 1, 0 ]/1  [ 2, 0 ]     3    3       5
+v^2        2      6   [ 2, 1 ]/1  [ -1, -3 ]  3    6       2
+v^2        2      5   [ 2, 1 ]/1  [ 3, 1 ]     3    6       6
+3v^2       2      4   [ 2, 1 ]/1  [  2, -2 ]  5    6       6
+v^2+v^3    3      1   [ 2, 1 ]/1  [  1, -3 ]  5    7       1
+v^2+v^3    3      0   [ 2, 1 ]/1  [  3, -1 ]  5    7       7
+v^2+v^3    3      3   [ 3, 0 ]/1  [ -2, -4 ]  3    9       3
+v^2+v^3    3      2   [ 3, 0 ]/1  [ 4, 2 ]     3    9       7
+2v^2+v^4   3      1   [ 3, 0 ]/1  [  0, -4 ]  5    9       2
+2v^2+v^4   3      0   [ 3, 0 ]/1  [ 4, 0 ]     5    9       8
+v^3+2v^4   2      4   [ 3, 2 ]/1  [  3, -3 ]  7    10      8
```

```
atlas> show_long (hodge_branch_std(trivial(G),10),x)
c         codim  x   lambda      hwt          dim  height  mu
+1        0      10  [ 2, 1 ]/1  [ 0, 0 ]     1    0       0
+1        2      4   [ 1, 0 ]/1  [  1, -1 ]   3    2       4
+1        3      3   [ 1, 0 ]/1  [ -2, -2 ]   1    3       2
+1        3      2   [ 1, 0 ]/1  [ 2, 2 ]     1    3       4
+2        3      1   [ 1, 0 ]/1  [  0, -2 ]   3    3       1
+2        3      0   [ 1, 0 ]/1  [ 2, 0 ]     3    3       5
+1        2      6   [ 2, 1 ]/1  [ -1, -3 ]   3    6       2
+1        2      5   [ 2, 1 ]/1  [ 3, 1 ]     3    6       6
+2+v      2      4   [ 2, 1 ]/1  [  2, -2 ]   5    6       6
+1+v      3      1   [ 2, 1 ]/1  [  1, -3 ]   5    7       1
+1+v      3      0   [ 2, 1 ]/1  [  3, -1 ]   5    7       7
+2v       3      3   [ 3, 0 ]/1  [ -2, -4 ]   3    9       3
+2v       3      2   [ 3, 0 ]/1  [ 4, 2 ]     3    9       7
+3v       3      1   [ 3, 0 ]/1  [  0, -4 ]   5    9       2
+3v       3      0   [ 3, 0 ]/1  [ 4, 0 ]     5    9       8
+2v+v^2   2      4   [ 3, 2 ]/1  [  3, -3 ]   7    10      8
```

19.4 $SL(2,\mathbb{C})$

Here is the tempered spherical representation of $SL(2,\mathbb{C})$:

```
atlas> G:=SL(2,C)
Value: connected quasisplit real group with Lie algebra 'sl(2,C)'
atlas> set q=all_parameters_gamma (G,[0,0])[0]
Variable q: Param
atlas> infinitesimal_character (q)
Value: [ 0, 0 ]/1
atlas> show_long (hodge_branch_std(q,20))
c      codim  x  lambda        hwt         dim  height  mu
+1     1      0  [ 0, 0 ]/1    [ -1,  1 ]  1    0       1
+v     1      0  [ 1, 1 ]/1    [ 0, 2 ]    3    2       2
+v^2   1      0  [ 2, 2 ]/1    [ 1, 3 ]    5    4       3
+v^3   1      0  [ 3, 3 ]/1    [ 2, 4 ]    7    6       4
+v^4   1      0  [ 4, 4 ]/1    [ 3, 5 ]    9    8       5
+v^5   1      0  [ 5, 5 ]/1    [ 4, 6 ]    11   10      6
+v^6   1      0  [ 6, 6 ]/1    [ 5, 7 ]    13   12      7
+v^7   1      0  [ 7, 7 ]/1    [ 6, 8 ]    15   14      8
+v^8   1      0  [ 8, 8 ]/1    [ 7, 9 ]    17   16      9
+v^9   1      0  [ 9, 9 ]/1    [  8, 10 ]  19   18      10
+v^10  1      0  [ 10, 10 ]/1  [  9, 11 ]  21   20      11
```

Here is the irreducible principal series with infinitesimal character ρ:

```
atlas> p
Value: final parameter(x=0,lambda=[1,1]/1,nu=[0,0]/1)
atlas> infinitesimal_character (p)
Value: [ 1, 1 ]/1
atlas> infinitesimal_character (p)=rho(G)
Value: true
atlas> composition_series (p)
Value:
1*parameter(x=0,lambda=[1,1]/1,nu=[0,0]/1) [2]
atlas> show_long (hodge_branch_std(p,20))
c     codim  x  lambda        hwt          dim  height  mu
+1    1      0  [ 1, 1 ]/1    [ 0, 2 ]     3    2       2
+v    1      0  [ 2, 2 ]/1    [ 1, 3 ]     5    4       3
+v^2  1      0  [ 3, 3 ]/1    [ 2, 4 ]     7    6       4
+v^3  1      0  [ 4, 4 ]/1    [ 3, 5 ]     9    8       5
+v^4  1      0  [ 5, 5 ]/1    [ 4, 6 ]     11   10      6
+v^5  1      0  [ 6, 6 ]/1    [ 5, 7 ]     13   12      7
+v^6  1      0  [ 7, 7 ]/1    [ 6, 8 ]     15   14      8
+v^7  1      0  [ 8, 8 ]/1    [ 7, 9 ]     17   16      9
+v^8  1      0  [ 9, 9 ]/1    [  8, 10 ]   19   18      10
+v^9  1      0  [ 10, 10 ]/1  [  9, 11 ]   21   20      11
```

19.5 $Sp(4,\mathbb{C})$

Here is the Hodge filtration on the spherical oscillator representation of $Sp(4,\mathbb{C})$. This is a ladder representation, with K-types in a line of multiplicity one. The Hodge filtration is given by the filtration on $U(\mathfrak{g})$, acting on the spherical vector.

```
atlas> G:=Sp(4,C)
Value: connected quasisplit real group with Lie algebra 'sp(4,C)'
atlas> rank(G)
Value: 4
atlas> rho(G)
atlas> {Here is the spherical oscillator representation}
atlas> p
Value: final parameter(x=7,lambda=[2,1,2,1]/1,nu=[3,1,3,1]/2)
atlas> infinitesimal_character (p)
Value: [ 3, 1, 3, 1 ]/2
atlas> dimension(LKT(p))
Value: 1
atlas> set h=hodge_branch_irr (p,50)
Variable h: ([ParamPol],void)
atlas> show_long (h)
c     codim  x  lambda            hw                  dim  height  mu
+v    4      0  [ 0, 0, 0, 0 ]/1  [ -2, -1,  2,  1 ]  1    0       7/1
+v^2  4      0  [ 1, 0, 1, 0 ]/1  [ -1, -1,  3,  1 ]  10   6       10/1
+v^3  4      0  [ 2, 0, 2, 0 ]/1  [  0, -1,  4,  1 ]  35   12      13/1
+v^4  4      0  [ 3, 0, 3, 0 ]/1  [  1, -1,  5,  1 ]  84   18      16/1
+v^5  4      0  [ 4, 0, 4, 0 ]/1  [  2, -1,  6,  1 ]  165  24      19/1
+v^6  4      0  [ 5, 0, 5, 0 ]/1  [  3, -1,  7,  1 ]  286  30      22/1
+v^7  4      0  [ 6, 0, 6, 0 ]/1  [  4, -1,  8,  1 ]  455  36      25/1
+v^8  4      0  [ 7, 0, 7, 0 ]/1  [  5, -1,  9,  1 ]  680  42      28/1
+v^9  4      0  [ 8, 0, 8, 0 ]/1  [  6, -1, 10,  1 ]  969  48      31/1
atlas>
```

References

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[2] Wilfried Schmid and Kari Vilonen, *Hodge theory and unitary representations of reductive Lie groups*, Frontiers of mathematical sciences, 2011, pp. 397–420. MR3050836

[3] ———, *Hodge theory and unitary representations*, Representations of reductive groups, 2015, pp. 443–453. MR3495806